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THE

MATHEMATICAL MONTHLY.

EDITED BY

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THE

MATHEMATICAL MONTHLY.

Vol. II....OCTOBER, 1859 ...No. I.

PRIZE PROBLEMS FOR STUDENTS.

I. SOLVE the equations

$$\sqrt{x} - 4 = \frac{259 - 10x}{4 + \sqrt{x}}$$

$$\frac{10}{x} - \frac{14 - 2x}{x^2} = 2\tfrac{4}{9}.$$

II. In any plane triangle, prove that the sines of the angles are inversely as the perpendiculars let fall from them upon the opposite sides.

III. Having given the diagonals of a quadrilateral inscribed in a given circle to determine its sides geometrically, when the diagonals intersect each other at right angles.

IV. Given

$$(1)\ xw + yz = ab,\ (2)\ xy + zw = ad$$

$$(3)\ xz + yw = bd,\ (4)\ x^2 + w^2 = y^2 + z^2;$$

to find the values of x, y, z, and w

V. If, in a plane or spherical triangle, A, B, C denote the angles, and a, b, c the opposite sides respectively; if r, ϱ denote the radii

of the circumscribed and inscribed circles, and δ the distance between the centres of these circles; then in the plane triangle

$$*\sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B \cos \tfrac{1}{2} C$$

and in the spherical triangle

$$\sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B \cos \tfrac{1}{2} C \frac{\cos \delta}{\cos r \cos \varrho}.$$

The solution of these problems must be received by the first of December, 1859. [See Editorial Items.]

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. IX., Vol. I.

The first Prize is awarded to William C. Cleveland, of the Lawrence Scientific School, Cambridge, Mass.

The second Prize is awarded to John W. Jenks, Senior Class, Columbia College, N. Y.

Prize Solution of Problem I.

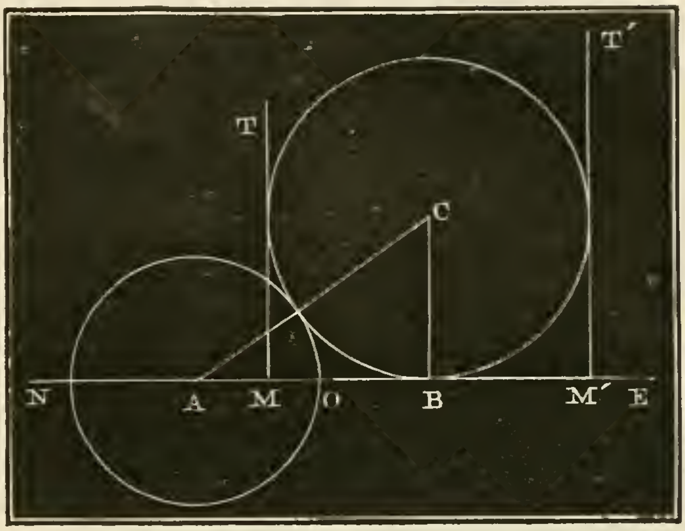
BY ASHER B. EVANS, MADISON UNIVERSITY, HAMILTON, N. Y.

"In a right-angled triangle, having given the difference between the base and perpendicular, and also the difference between the hypothenuse and base; to construct the triangle geometrically."

On the indefinite line NE take $AM = a =$ the difference between the base and perpendicular, and erect the perpendicular MT. Also, with A as a centre, and $AO = b + a =$ he difference

* This analogy, pointed out by D'Arrest, is only one of a number of interesting ones given by Professor Chauvenet, in Gould's Astronomical Journal, Vol. III., page 50, without demonstration. We propose to give them from time to time in our lists of Prize Problems, as most valuable exercises in Trigonometry.

between the hypothenuse and perpendicular as radius, describe a circle. *Now if C be the centre of a circle tangent to this circle, and also to the sides of the rightangle $T\,M\,E$ then will $A\,B\,C$ be the required triangle. For, $A\,B = B\,C + A\,M = B\,C + a$, and $A\,C = B\,C + A\,O = B\,C + b + a = A\,B + b$.



Prize Solution of Problem II.

BY ASHER B. EVANS, MADISON UNIVERSITY, HAMILTON, N. Y.

"In a right-angled triangle, having given the sum of the base and perpendicular, also the sum of the hypothenuse and base; to construct the triangle geometrically."

Let x be the perpendicular, $a - x$ the base, and $b - a + x$ the hypothenuse. On the indefinite line NE take $AM' = a$, and draw the perpendicular $M'\,T'$. Also, with A as a centre, and $A\,O = b - a$ as a radius, describe a circle. If C be the centre of a circle tangent to this circle, and also to the sides of the right angle $A\,M'\,T'$, then will $A\,B\,C$ be the required triangle. For, $A\,B = A\,M - B\,C = a - x$, and $A\,C = A\,O + B\,C = b - a + x$.

Second Solution of Problems I. and II.

BY GEORGE A. OSBORNE, JR., LAWRENCE SCIENTIFIC SCHOOL.

Problem I. Denote by a the difference between the hypothenuse and base, and by b the difference between the perpendicular and

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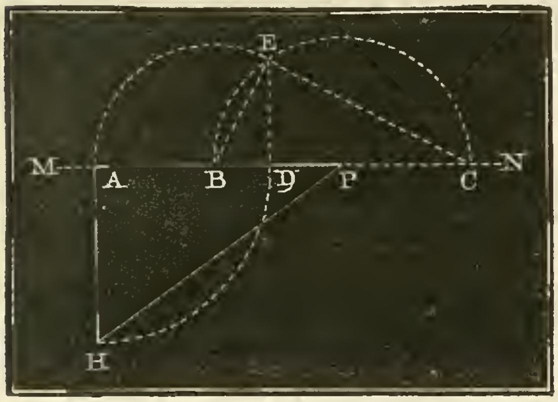
* To draw a circle tangent to a given circle, and to the sides of a rightangle, we may employ Problem 8, in a Memoir on the "Tangencies of Circles," by Major Benjamin Alvord (Smithsonian Contributions to Knowledge); or a Theorem of Professor H. A. Newton, found on page 242 of the Mathematical Monthly, Vol. I.

base; also, let $x =$ base, $y =$ perpendicular; therefore, $\sqrt{x^2+y^2} =$ hypothenuse.

There may be two cases, according as the perpendicular is longer, or shorter, than the base.

For the first case, we have $\sqrt{x^2+y^2} - x = a$, and $y - x = b$; by eliminating y, we readily obtain $x = a - b \pm \sqrt{2a(a-b)}$; rejecting the negative sign before the radical, as giving a negative value to x, we have $x = a - b + \sqrt{2a(a-b)}$ and $y = x + b$.

To construct these values of x and y: From any point P on the line, MN, with radius $PB = a$, describe a semicircle. Lay off $PD = b$, erect the perpendicular DE, make $AB = BE$, and AH equal and perpendicular to AD. Joining H with P forms the required triangle, PAH.



For $AB = BE = \sqrt{BD.BC} = \sqrt{2a(a-b)}$. Therefore, $AH = AD = BD + AB = a - b + \sqrt{2a(a-b)} = x$, and $AP = AD + PD = x + b = y$.

If $b > a$, the value of x will be imaginary, and the construction impossible.

For the second case, $x - y = b$, or $y - x = -b$; hence, if in the expressions for x and y, in the first case, we make b negative, they will apply for the second case; namely,

$$x = a + b + \sqrt{2a(a+b)} \text{ and } y = x - b.$$

If $b < a$, the triangle may be constructed as in the first case, provided we make the same construction with respect to C and CE, as was there made with respect to B and BE.

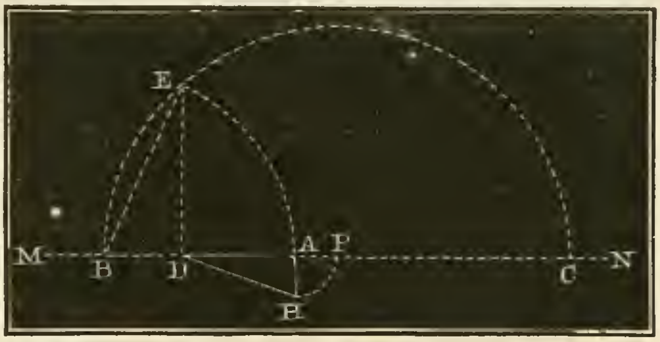
If $b > a$, the semicircle must be described on $a + b$ as a diameter, instead of on $2a$, as before.

Problem II. Denote by a the sum of the hypothenuse and base, and by b the sum of the base and perpendicular; also let $x =$ base, $y =$ perpendicular; therefore, $\sqrt{x^2+y^2} =$ hypothenuse.

Hence, $\sqrt{x^2+y^2}+x=a$, $x+y=b$. Eliminating y gives $x = -(a-b) \pm \sqrt{2a(a-b)}$; rejecting the negative sign before the radical as giving a negative value to x, we have

$$x = -(a-b)+\sqrt{2a(a-b)} \text{ and } y=b-x.$$

To construct these values of x and y: From any point P on the line MN with radius $PB=a$, describe a semicircle. Lay off $PD=b$, erect the perpendicular DE, make $BA=BE$ and AH equal and perpendicular to AP. Joining H with D forms the required triangle DAH.



For $BA=BE=\sqrt{BD.BC}=\sqrt{2a(a-b)}$ $\therefore$ $AD=BA-BD$ $=\sqrt{2a(a-b)}-(a-b)=x$. $AH=AP=PD-AD=b-x=y$.

If $x>b$, or $\sqrt{2a(a-b)}>a$, the value of y will be negative, and the construction impossible; hence, $\sqrt{2a(a-b)}<a$, or $BA<BP$, and the point A always falls between B and P. The expression $\sqrt{2a(a-b)}<a$ is equivalent to $b>\frac{1}{2}a$.

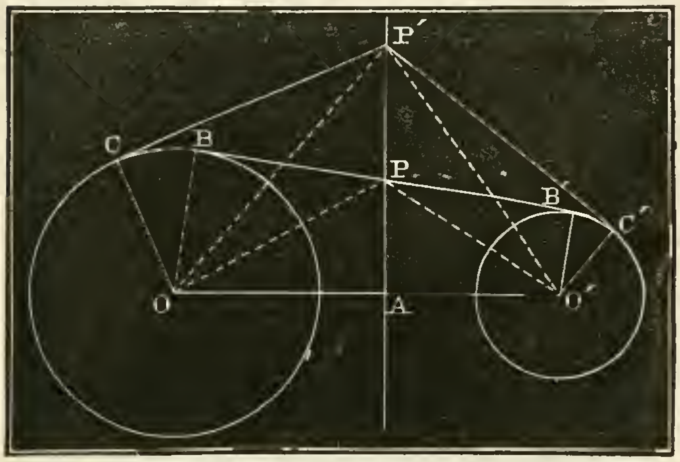
Prize Solution of Problem III.

by George A. Osborne, Jr., Lawrence Scientific School.

"The four tangents, which are common to two circles which do not intersect, and are terminated at their points of respective contact, have their middle points on the radical axis of the two circles."

Through P, the middle point of the common tangent BB', draw $P'A$ perpendicular to OO'. From any point P' of this line draw the tangents $P'C$ and $P'C'$; draw also PO, PO', $P'O$, and $P'O'$. Then we have $P'O^2 - OC^2 = P'C^2$ and $PO^2 - OB^2 = PB^2$.

Therefore, by subtraction $P'O^2 - PO^2 = P'C^2 - PB^2$. Similarly we find $P'O'^2 - PO'^2 = P'C'^2 - PB'^2$. Also, $P'O^2 - PO^2 = P'A^2 - PA^2 = P'O'^2 - PO'^2$. $\therefore P'C^2 - PB^2 = P'C'^2 - PB'^2$. But $PB = PB'$; $\therefore P'C = P'C'$. Hence, if from any point of the line $P'A$ tangents be drawn to the two given circles, they will be equal. *$P'A$ is therefore the radical axis of the two circles, and will bisect the common tangents.

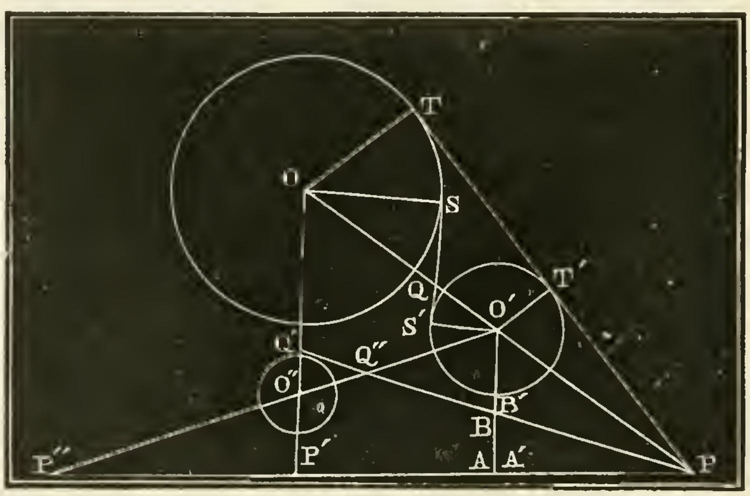


PRIZE SOLUTION OF PROBLEM IV.

BY JOHN W. JENKS, SENIOR CLASS, COLUMBIA COLLEGE, N. Y.

"The external centres of similitude of three circles, taken successively two and two, all lie in one straight line; and each of them is situated in a right line with two of the internal centres of similitude."

In any two circles (O, O') the distances $(OP, O'P)$ from the centres to the external centre of similitude (P) are by similar triangles $(OTP, O'T'P)$ proportional to the radii of the two circles. By like similar triangles $(OSQ, O'S'Q)$ the distances from the centres to the internal centre of similitude are also seen to be proportional to the radii.



* This Problem is a special case of Proposition 5, on page 270, of the June number of the MONTHLY, as was remarked by nearly all the contributors.

Let O O', O'' be the three given circles of the problem, and denote their radii by R, R', R'' respectively; also let P be the external and Q the internal centre of similitude of O and O', P' and Q' of O and O'', and P'' and Q'' of O' and O''. Then will P, P' and P'' lie in one straight line, as also Q', Q'' and P, also Q, Q'' and P', and Q, Q' and P''.

For, draw the lines $O\,O'P$, $O\,O''\,P'$ and $O'\,O''\,P''$, also $P''\,P'$ and $P\,P'$. Now, drawing $O'\,A'$ parallel to $O\,O''\,P'$ to meet $P''\,P'$ produced in A' we have by similar triangles $O'A' : O''P' = O'P'' : O''P'' = R' : R''$. $\therefore$ $O'A' = \frac{O'\ P' \times R'}{R''}$. Drawing also $O'A$ parallel to $O\,O''P'$ to meet $P\,P'$ in A, we have, by similar triangles, again, $O'A : O\,P' = O'\,P : O\,P = R' : R$ $\therefore$ $O'\,A = \frac{O\,P' \times R'}{R}$. But $O\,P' : O''\,P' = R : R''$, or $O\,P' : R = O''\,P' : R''$ $\therefore$ $\frac{O\,P' \times R'}{R} = \frac{O''\,P' \times R'}{R''}$. $\therefore$ $O'\,A = O'\,A'$. Hence the two lines $P''P'$ and $P\,P'$, having two points in common (P' and A), coincide throughout their whole extent, and form one and the same straight line.

In a precisely similar manner, by drawing the line $P\,Q''$ and $Q'\,Q''$ and drawing $O'B$ to $P\,Q'$ parallel to $O\,O''P'$, and $O'B'$ to $Q'\,Q''$, produced, parallel to the same, we have $O'\,B : O\,Q' = O'\,P : O\,P = R' : R$. $\therefore$ $O'\,B = \frac{O\,Q' \times R'}{R}$; also $O'\,B' : O''\,Q' = O'\,Q'' : O''\,Q'' = R' : R''$. $\therefore$ $O'\,B' = \frac{O''\,Q' \times R'}{R''}$.

But $O\,Q' : O''\,Q' = R : R''$. $\therefore$ $O\,Q' : R = O''\,Q' : R''$. $\therefore$ $O'\,B = O'\,B'$. Hence $P\,Q''$ and Q' are in the same straight line. In the same way it may be proved that $P'\,Q''$ and Q, and also $P''\,Q'$ and Q, are in the same straight line with one another.

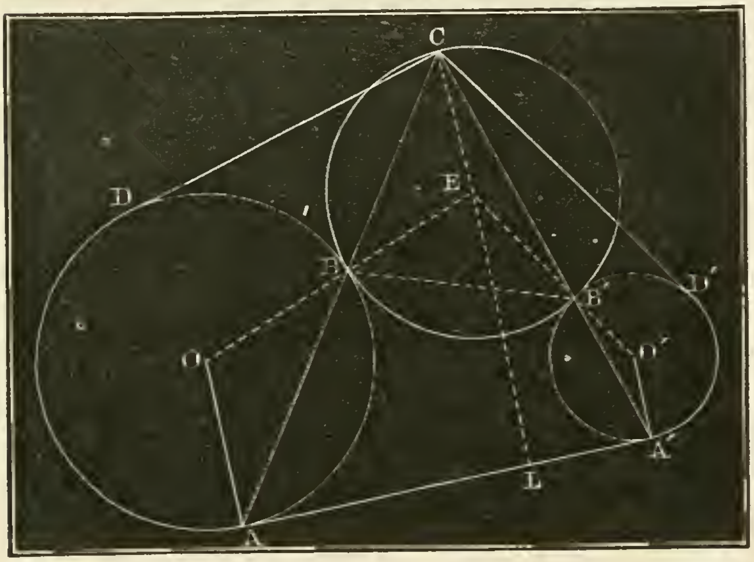
Prize Solution of Problem V.

BY WILLIAM C. CLEVELAND, LAWRENCE SCIENTIFIC SCHOOL.

" Let two circles be touched respectively by a single straight line $A\,A'$ in A and A'.

and by a single circle in $BB'C$ in B and B'; if the straight line and the circle touch in the same manner the two circles, the point C of the meeting of AB and $A'B'$ will lie on the circumference of the circle $BB'C$ and on the radical axis of the two other circles."

Let E be the centre of the circle $BB'C$, and O and O' the centres of the two other circles. The line OE passes through B, and $O'E$ through B'. OA and $O'A'$ are parallel, since they are both perpendicular to AA'. Through E draw EL parallel to OA, and suppose AB produced to meet EL produced in C, and $A'B'$ to meet in C'. The similar triangles AOB and BCE give $\frac{EC}{EB} = \frac{AO}{OB} = 1$, and $A'O'B'$ and $B'C'E$ give $\frac{EC'}{EB'} = \frac{O'A'}{O'B'} = 1$. $\therefore EC = EB = EB' = EC'$.



Therefore C and C' are the same point on the circumference of $BB'C$.

Draw the tangents CD and CD'; then

$$\frac{CD^2}{CD'^2} = \frac{CA \cdot CB}{CA' \cdot CB'} = \frac{CA \cdot 2\,EB \cos ECB}{CA' \cdot 2\,EB' \cos ECB'} = \frac{CA \cos ECB}{CA' \cos ECB'} = \frac{CL}{CL} = 1.$$

Therefore $CD = CD'$, and C is on the radical axis of the circles O and O'.

Chauncey Wright.
Truman Henry Safford.

Note. — Prof. Winlock has been ordered to the Naval Academy at Annapolis, Maryland, as Professor of Mathematics.

MECHANICAL CONSTRUCTION OF THE AREA OF A CIRCLE.*

By Chauncey Smith, of the Suffolk Bar, Boston.

Proposition. *The area of a circle is equal to the rectangle contained by its semicircumference and semidiameter.*

Children usually find this proposition in their arithmetics, in the form of a rule, before they have any knowledge of geometry; and I therefore propose the following simple, graphic method of demonstration.

Cut a circle from a piece of card, and bisect it as in Fig. 1. Divide each half into an equal number of sections, cutting from the centre, *C*, but leaving them all joined together at the circumference. Next,

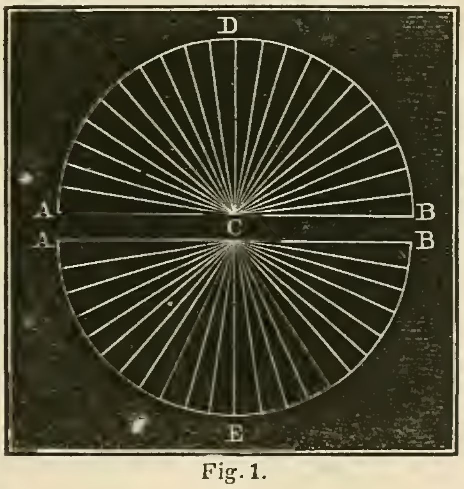


Fig. 1.

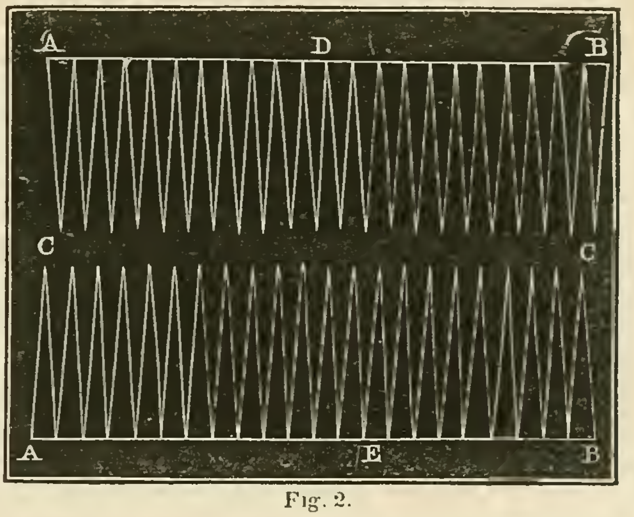


Fig. 2.

draw the points *A* and *B*, as nearly as may be, into straight lines with *D* and *E*, as in Fig. 2. Lastly, slide the halves together as

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* This graphic method may be exhibited equally well upon the blackboard, by drawing the figures, and showing how one is made from the other; and it will be found of great advantage in those cases in which the pupil is called upon to work by an arbitrary rule, as in most of the arithmetics. To the same end, the rectangle may be divided into squares, and the pupil made to see that if one of these squares is taken as the unit of surface, that the surface of the whole rectangle will be measured by the product of its base by its altitude; that is, this product will show how many such units the rectangle contains. Indeed, the

in Fig. 3. If the number of sections is very great, ADB and AEB in Fig. 2 will be very nearly straight lines, and Fig. 3 will differ but little from a rectangle. Its surface is clearly the same as that of the circle out of which it was made; but its base and altitude are the semi-circumference and semidiameter of the circle, and their product is the measure of its area, which is therefore the measure of the area of the circle.

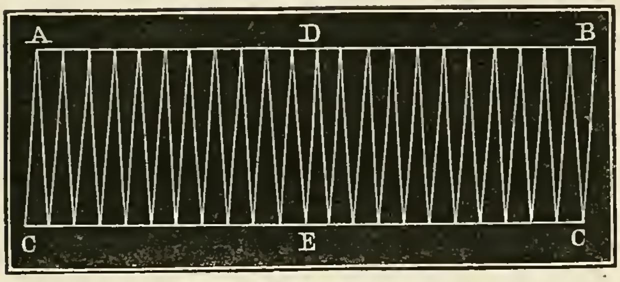

Fig 3.

ON THE PRIME SEVENTH AS AN ESSENTIAL ELEMENT IN THE MUSICAL SYSTEM.

By Henry Ward Poole, Engineer, Boston.

It is now ten years, since, by original investigation in the mathematical, mechanical, and practical departments of music, I was led to the belief that this science has a solid foundation in the relations of numbers, and that all the supposed impossibility of Just Intonation and the necessity of Temperament, have their origin only in the short-sightedness of the theorist, and the unskilfulness of the practitioner.

Having settled upon the rule that musical ratios must not exceed a certain limit of simplicity (the limit to be determined by the

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areas of all the surfaces investigated in plane geometry may be exhibited to the eye by simple construction; and quite young children may readily be taught to understand as well as make the constructions for themselves. — Ed.

Since receiving the above, the same construction has been sent us by R. C. Matthewson, Esq., of San Francisco, California; and we also find it in Lund's Geometry, an English book of recent date.

ability of the ear to appreciate them), it was stated* that those ratios only were admissible which were derived from the prime numbers 2, 3, 5, and 7. That the three lower primes 2, 3, and 5 belong to the musical system has been universally admitted; but no one, before myself, so far as I know, has made this claim for the prime seven.†

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* American Journal of Science, Second Series, Vol. IX. pp. 68, 199.

† I do not wish to conceal the fact, that even now the principle of just intonation (or the possibility, in theory or practice, of exact fifths, thirds, &c.) is denied by high mathematical authorities. Sir J. F. W. HERSCHEL, in his treatise on Sound, declares, that "singers, violin players, and all others who can pass through every gradation of tone, must all temper, or they could never keep in tune with each other or themselves." [The work of HERSCHEL not being at hand, this extract is copied from the treatise on *Sound*, by Professor BENJAMIN PEIRCE, of Cambridge, who has reproduced (with his indorsement, it is presumed,) these and like views of HERSCHEL.] By a late letter from Sir JOHN HERSCHEL, dated Collingwood, June 14th, 1859, addressed to the Musical Pitch Committee, at the Society of Arts, he evinces his continued belief in Temperament as inherent in music, and his opinion that this temperament gives some peculiar character to the different signatures or keys in music in general. He says, in regard to the concert pitch: —

"All are desirous that when once lowered, it should be kept from rising [1] again, to which there is a continual tendency arising from a distinct natural cause inherent in the nature of harmony; namely, the excess (amounting to about eleven vibrations in ten thousand) of a perfect fifth over seven-twelfths of an octave, which has to be constantly contended against in upward modulations, whenever violins or voices are not kept in check by fixed instruments. But perhaps all are not aware that the evil of fine ancient compositions having thus been rendered impracticable to singers in their original normal key involves the sacrifice of the adaptation of the peculiar character of the key (a character intended and felt by the composer), and the substitution of a totally different incidence of the temperament [2] in the series of notes in the scale, and goes therefore to mar the intended effect, and injure the composition, as much as an ill-chosen tone of varnish would damage the effect of a fine Titian."

1. There is nothing better to test the "natural tendency" in this respect than a good glee-club without accompaniment. If a high pitch is taken and they are fatigued, the pitch will gradually fall. If they start with too low a key-note and are in good spirits, the tendency will be to rise to the better pitch. It does not appear that temperament affects the concert pitch.

2. Observe the same glee-singers. They sing in every key with the same relative intervals, and *do not* use a "different incidence of temperament," in different keys. Did any composer of glees wish such temperament? If so he should indorse his score something after this manner: "Four flats, equal temperament" (as the composers of fugues for the organ have actually done;) or "Four flats, with a great wolf in A flat, and a whelp in E flat."

I only desire here to put on record for historical reference the most respectable authorities of this day against Just Intonation, and to prove that the views I put forth have such opponents, and hence need to be told.

The interval 4 : 7 derived from the prime seventh has not been unnoticed, as a curiosity in acoustics; and it is occasionally referred to as the "Za" of Tartini. A living writer,* whose statements are entitled to the highest respect, and whose works contain most able arguments in favor of Just Intonation, says of the sounds produced from the prime seventh: "They may be called *anomalous.* They are wheels, but not wheels which will fit in with the previously constructed parts of the machine, and therefore they are left on one side."

The sound 4 : 7 has been known to be the seventh harmonic of the horn and æolian string, but has been called a "false" note, and has been rejected even by the advocates of just intonation, as opening the way for inextricable complication in theory and practice. It will from this, appear necessary to make the declaration which is the subject of this paper, and which is as follows:

The Prime Seventh belongs to the Musical System; its ratios are altogether appreciable by the common ear, and are in constant use in common music. It is this which constitutes, when added to the common chord, the *concord* (falsely called the *discord*) of *the Seventh,* and this element, combined with the other prime chords of Octave, Fifth, and Major-Third, makes the great variety of noble harmonies in which cultivated and uncultivated ears delight.

The prime seventh is necessary to complete the series of simple ratios, which extend as far as 10; and it was by noticing the blanks which its omission would leave that its necessity was discovered. The series is as follows:

1 : 2, 2 : 3, 3 : 4, 4 : 5, 5 : 6, [6 : 7, 7 : 8,] 8 : 9, 9 : 10

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* Gen. T. Perronet Thompson. Just Intonation. p. 72. 2d Edition. London, 1857. See also his "*Exercises, Political and Others.*" London. 1843. 6 vols. Both are in the Boston Athenæum.

or, if written as below, we shall have the natural series of harmonics, or what may be called the primary or

HARMONIC SCALE.

$$1:2:3:4:5:6:7:8:9:10.$$

As some reason should be assigned to the mathematician for not extending the series by the introduction of the *Prime Eleventh*, it will be found in the inability of the human ear to appreciate such complicated relations. The "*Chord of the Eleventh*" exists in nature, and I am able to tune it and to recognize its harmony in combinations specially made for the experiment; yet, so far as my examination of the works of the masters has extended, it has not been used by them in their written music, and perhaps never, unless possibly in the harmonics of a Paganini. I do not claim for it a place in our practical system of music, but leave it where all the former theorists have set the Prime Seventh. The Eleventh may hereafter be admitted, when the musical faculties of men have been sharpened by familiarity with the more simple concords *in their purity*, and when music is carried to a higher degree of refinement.

From the *Harmonic Scale* may be derived, by combination of its chords, an indefinite variety of other scales. The Octave is divided into eight intervals, which are convenient for melodic use, and the result is popularly called the Diatonic Scale, which, although generally taken as the basis, in explaining music, is not a primary, but a secondary Scale. The method of forming it, according to all former treatises, is, to take common chords (4 : 5 : 6) upon the tonic, dominant, and subdominant. Thus, the scale of C is tuned by taking a Fifth and Major-Third on C, on G, and on F, and bringing all the notes within the same octave.

But the introduction of the prime seventh allows of another division, in which only two fundamentals are employed; namely, the

Tonic (E) with its common chord, (C, E, G) and the Dominant (G), on which is taken the chord of seventh and ninth (G, B, D, F, A) in the ratios 4 : 5 : 6 : 7 : 9. To distinguish these scales, I have called the first the *Triple Diatonic*, and the last the *Double Diatonic*. Assuming the tonic or the key-note, as C, with 48 vibrations, the two scales will stand as follows:—

TRIPLE DIATONIC.

(With common Chord on C, on G, and on F.)

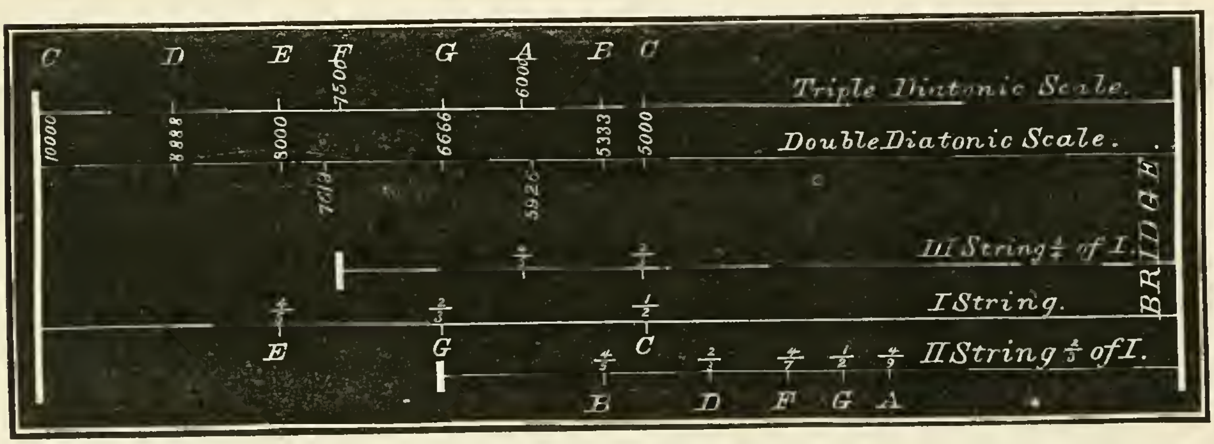
C	D	E	F	G	A	B	C
48	54	60	64	72	80	90	96
8 : 9	9 : 10	15 : 16	8 : 9	9 : 10	8 : 9	15 : 16	

DOUBLE DIATONIC.

(With common Chord on C: and Chord of 7 and 9 on G.)

C	D	E	(F)	G	(A)	B	C
48	54	60	63	72	81	90	96
8 : 9	9 : 10	20 : 21	7 : 8	8 : 9	9 : 10	15 : 16	

THE FOLLOWING DIAGRAM EXHIBITS THE MONOCHORD, WITH THE DERIVATION OF THE TWO DIATONIC SCALES, AND THE PROPORTIONAL LENGTHS OF THEIR STRINGS OR PIPES, THE PRIMARY HAVING 10000 EQUAL PARTS.



It thus appears that the fourth and sixth notes may be taken differently in intonation; and that this is done, can be easily observed by giving attention to singers. The Triple Diatonic has but three

different intervals; namely, 8 : 9*, 9 : 10 and 15 : 16. The Double Diatonic has, in addition, two others; namely, 20 : 21*, and 7 : 8; and in combinations its variety is greatly superior to the Triple Scale, whose chords and intervals are rather duplicates of one another.

And the remarkable fact is, that this Double Diatonic, which no theorist has defined, is more in practical use than the Triple, which stands in all the elementary books. A familiar example of the former is the "*O dolce Concento*" of MOZART, and the principal movement of the "*Dead March in Saul*" of HANDEL. The melody of the "*Hundredth Psalm*" is in the Triple Diatonic. The two scales often interchange, and an example of this is to be found near the close of "*O dolce Concento*," where for a single measure the dominant seventh and ninth yield, to admit the fourth and sixth of the Triple Scale.†

If it be feared that the distinctions which have been described as belonging to the scale will complicate it for those learning to sing or play, let it be added, that singers naturally observe them all, and need have no other instruction than to hear the sounds given by their teacher. What is here set down is of interest to him who wishes to know what is, and what ought to be, done. It may not be necessary for the singer to be even told the dimensions of any of his

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* In view of the numerous names required, and to supply those needed for these unnamed intervals, I have proposed (at least for mathematical and theoretical uses) names derived from the ratio. A fifth then will be "two-three," a major-tone "eight-nine," a diatonic semitone "fifteen-sixteen," the interval, (unnamed) between the third of the scale and the dominant seventh, is the "twenty-twenty-one." The next interval (dom. seventh to fifth) is the "seven-eight." This proposition as yet needs the approval of other theorists. The desideratum is accuracy and clearness.

† Not to disfigure these mathematical pages with musical types, I have chosen examples with which every one is familiar. Every composition will furnish others. If a choice is to be made in the examples of the profuse employment of the prime seventh, there may be taken any of the vocal scores of HAYDN, MOZART, or ROSSINI.

intervals; and it perhaps does no harm (except to the one who utters the falsehood) to say that all intervals are compounded of "semitones" or artificial twelfths of the octave.

It is true that when all the four primes have furnished their numerous chords and intervals, we shall have assembled a large number of notes, and it is not impossible that those unacquainted with music may fear that the number will be unmanageable, and prefer the compromises and limitations of temperament. As the experiment has been practically made, such persons may be assured that the musician can most easily produce his desired effects, when he has the full and abundant materials which the system of just intonation gives him.

The singers and players upon the free instruments, of their own accord, use the true intervals to the best of their ability; and in spite of the tempered instruments with which they are sometimes obliged to join. It is for men of science to indicate to the makers of imperfect instruments the way to perfect them; and to withhold their approval from players, who, from indolence or incapacity, only make a pretence of interpreting the music of the great masters. There are wanted no more apologies for, or speculations upon, the choice of temperaments; that subject has long ago been exhausted; and nothing more can be done than is now done with twelve tempered notes in the octave. When some economical astronomer shall propose to reduce the bulk and expense of the Nautical Almanac, by sacrificing that accuracy which gives it priceless value, the men at Greenwich will regard him as the scientific musician will, at a future day, look on those who would restrict him to the meagre and barbarous systems of temperaments of twelve notes.

NOTES AND QUERIES.

1. *Least Common Multiple.* On page 396, Vol. I., it is asked, whether the rule for finding the least common multiple, which says, divide by *any* number, will give the same result as the rule which says, divide by *any prime* number. My answer is, not always. If we divide the given numbers by *any prime* number which will divide two or more of them without a remainder, we shall, by following the rule, always get their *least* common multiple; if we divide by *any* number which will divide two or more of them, and follow the rule, we shall get a *common* multiple, but not always the *least.* The reason is plain; but may be most clearly seen by means of an example.

Find the least common multiple of 30, 63, 66, 300.

First Solution.

	30	63	66	300
6	$2.3.5$	$3^2.7$	$2.3.11$	$3.2^2.5^2$
5	5	$3^2.7$	11	2.5^2
	1	$3^2.7$	11	2.5

$$6.5.3^2.7.11.2.5 = 207900$$

Second Solution.

	30	63	66	300
2	$2.3.5$	$3^2.7$	$2.3.11$	$3.2^2.5^2$
3	3.5	$3^2.7$	3.11	$3.2.5^2$
5	5	3.7	11	2.5^2
	1	3.7	11	2.5

$$2.3.5.3.7.11.2.5 = 69300$$

The first solution gives a result three times too large, as it should. For when we divided by 6, the factor 3 common to all the numbers was taken out of only three of them. The least common multiple of several numbers is composed of all the factors not common to them, and of all the factors common to two or more of them, and of no others. Any common factor raised to the highest power to which it is found in either of the given numbers, must enter into the least common multiple, otherwise the supposed multiple would not be divisible by this number. On the other hand,

if any common factor, raised to a higher power than it is found in any of the given numbers, enters into the common multiple, then such multiple will not be the least, as is seen from the first solution; since 3 raised to the third power is found in the multiple 207900, which is one higher than it is found in either of the given numbers.—TEACHER.

2. On page 363, Vol. I., we find that 17 horses were devised to three sons, as follows: To the first, $\frac{1}{2}$; to the second, $\frac{1}{3}$, and to the third, $\frac{1}{9}$; and in the distribution by the Judge, the first got 9, the second 6, and the third 2. The question is, Was the distribution just?

Now it is evident that the testator intended to give the whole of the 17 horses to his three sons; and this he thought he had done; when, in fact, according to the terms of the will, he had only disposed of $\frac{17}{18}$ths of the horses, there being $\frac{1}{18}$ undisposed of. For $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{9}$ only make $\frac{17}{18}$ths. But the rule is, to interpret wills so as to effectuate the *intention* of the testator, when that can be ascertained. In this case, it is plain he intended to give them *all* the horses, and named the proportions they should have. All that is necessary, then, to effectuate his intention, would be to say, As the whole portion of the horses actually devised to his three sons by the letter of the will is to the whole number of horses *intended* to be devised to them all, so is the particular proportion actually devised to any of the sons to the number or portion *intended* to be devised to that particular son.

Thus:

$$\tfrac{17}{18} : 17 :: \tfrac{1}{2} : 9 = \text{first son's part.}$$
$$\tfrac{17}{18} : 17 :: \tfrac{1}{3} : 6 = \text{second son's part.}$$
$$\tfrac{17}{18} : 17 :: \tfrac{1}{9} : 2 = \text{third son's part.}$$

So that the distribution is proved to be just, without resorting to the Judge's horse for aid in the interpretation of the will.

Or suppose a be the whole number of horses to be distributed. Then it will be as

$$\frac{17a}{18} : a :: \frac{a}{2} : \frac{9a}{17} = \text{the first son's part.}$$

$$\frac{17a}{18} : a :: \frac{a}{3} : \frac{6a}{17} = \text{the second son's part.}$$

$$\frac{17a}{18} : a :: \frac{a}{9} : \frac{2a}{17} = \text{the third son's part.}$$

So that a may be 17, or any number which is a multiple of 17. When that is the case, all the conditions of the question can be complied with, so far as the distribution is concerned, by multiplying the number of horses by 9, and dividing by 17, for the first son's part; by multiplying by 6, and dividing by 17, for the second son's part; and multiplying by 2, and dividing by 17, for the third son's part. — SAXE GOTHA LAWS, Attorney and Counsellor at Law, Dover, Delaware.

3. *Note on Maxima and Minima.* The following deduction of the rule for determining maxima and minima values of a function may aid the student of the Calculus to a better understanding of the common symbolic demonstration.

A function has a maximum or minimum value when its rate of change, or the first derivative, vanishes, and in vanishing, changes its sign.

Since a positive value in the derivative indicates an increase of the function for an increase of the variable, and a negative value indicates a decrease of the function for an increase of the variable; then a change in the derivative from positive to negative values indicates a change in the function from increase to decrease, and hence the value of the function is a maximum.

On the other hand, a change in the derivative from negative to positive values indicates a change in the function from decrease to increase, and the function has a minimum value.

There are four ways in which a continuous real quantity may vanish: 1st, by decreasing from positive to negative values; 2dly, by increasing from negative to positive values; 3dly, by having zero as a minimum value; and 4thly, by having zero as a maximum value. These four states of zero may be expressed by the symbols $+0-$, $-0+$, $+0+$, and $-0-$; in which the signs indicate the states of the quantity before and after vanishing.

If a derivative has, for any value of the variable, the value $+0+$, or $-0-$, then the function increases or decreases both before and after it has ceased to increase or decrease; that is, it simply suspends for a moment without reversing the change, and its value is therefore neither a maximum nor a minimum.

If a series of successive derivatives all vanish for a given value of the variable; and if the following one which does not vanish, have a positive value, then the last of the vanishing derivatives is zero by increase, and is therefore of the form $-0+$. The last but one is thence a minimum zero, or $+0+$; the last but two is $-0+$, the same as the last of the series. The next lower is, therefore, $+0+$, and so on.

On the other hand, if the value of the derivative which does not vanish be negative, then the last of the series will be $+0-$, the last but one $-0-$, the last but two $+0-$, and so on.

If we denote by D^n the derivative following the vanishing series, and by D^{n-1}, D^{n-2}, &c., the derivatives of the vanishing series, these propositions may be exhibited as follows:

If	$D^n = +a$	or	$-a$
then	$D^{n-1} = -0+$	or	$+0-$
	$D^{n-2} = +0+$	or	$-0-$
	. . . .	.	. .
	$D^{n-2i} = +0+$	or	$-0-$
and	$D^{n-2i-1} = -0+$	or	$+0-$

Hence if D^{n-2i} be the first derivative of the function, n must be $2i+1$, an odd number, and the value of the function is neither a maximum nor a minimum, since it does not reverse its rate of change. But if D^{n-2i-1} be the first derivative n must be $2i+2$, an even number, and the value of the function is a minimum when $D^n = +a$, and a maximum when $D^n = -a$. Hence the rule.

If the derivative, the first that does not vanish for a given value of the variable, be positive, and of an even order, the function is at a minimum value; if this derivative, be negative and even, the function is at a maximum value; but if this derivative be of an odd order, then the value of the function is neither a maximum nor a minimum. W.

ON THE INDETERMINATE ANALYSIS.

By Rev. A. D. Wheeler, Brunswick, Maine.

DEFINITIONS.

1. That branch of Algebra which treats of the solution of indeterminate or unlimited problems is denominated The Indeterminate Analysis.

2. An Indeterminate Equation is one that contains two or more unknown quantities.

3. The Diophantine Analysis is that in which the indeterminate quantities are required to be of the second, or of some higher degree.

4. The Factors of a number are any numbers which, multiplied together, will produce it; or in other words, they are any of its exact divisors.

5. Prime Numbers are those which have no factor, excepting unity and the numbers themselves.

6. The Prime Factors of a number are its least divisors.

7. Numbers are said to be prime to each other, when they have no common divisor.

8. All numbers that are not prime are termed composite numbers.

9. An Integer is a number composed of entire units.

AXIOMS.

1. If an integer be added to an integer, the sum will be an integer.

2. If an integer be subtracted from an integer, the remainder will be an integer.

3. If an integer be multiplied by an integer, the product will be an integer.

4. If an integer be divided by any of its factors, the quotient will be an integer.

PROPOSITIONS.

I. If a number will divide the sum of two or more numbers, and all the parts excepting one, then will it also divide the remaining part.

Demonstration. Let $S = A + B$. Then if d be an exact divisor of S and A, it will also be an exact divisor of B. For if we supposed d to be contained n times in S, and n' times in A, then will it obviously be contained $n - n'$ times in B; and this must be an integer according to Axiom 2.

In the same way may the truth of the proposition be shown, whatever be the signs, or the number of parts.

II. If a number will divide all the parts of any number excepting one, and will not divide that one, then it will not divide their sum.

Demonstration. If it would divide their sum, then, according to what has just been proved, it would likewise divide all the parts; which is contrary to the supposition.

III. If P be a prime number and will divide the product, $A.B$, then will it divide one of the factors A, or B.

Demonstration. Since P is a factor of the product, $A.B$, it must be found among the factors of either A or B, and consequently it must divide one or the other.

IV. If P will divide neither A nor B, it will not divide their product.

Demonstration. If it would divide their product, then, according to the last proposition, it must divide A or B; which is contrary to the supposition.

V. If a and b are prime to each other, and if, in the expression $\frac{by}{a}$, we assign to y every possible integral value less than a, and positive, the expression can never become an integer, and in no two cases shall we have the same remainder.

Demonstration. (1) As a is prime to b, and greater than y, it will divide neither factor, and consequently will not divide their product, Prop. IV.

(2). Suppose that any two values of y, as y' and y'', will give the same remainder r, then $\frac{by'-r}{a}$, and $\frac{by''-r}{a}$, will both be integers. Hence, also, their difference $\frac{by'-r}{a} - \frac{by''-r}{a} = \frac{b(y'-y'')}{a}$, (Axiom 2) is an integer. But as $y'-y''$ is less than y', which is, by hypothesis, less than a, it is plain that a will divide neither factor, and therefore it is impossible that $\frac{b(y'-y'')}{a}$ should be an integer. Consequently no two remainders can have the same value.

Cor. 1. Since the remainders are all different, and the num-

ber of them is equal to $a-1$, they must evidently comprehend the entire series of natural numbers between the limits 0, and a. Let any one of these remainders be r; then $\frac{by-r}{a}$ will be an integer; or let it be $a-r$; then $\frac{by-(a-r)}{a}=\frac{by-a+r}{a}=-1+\frac{by+r}{a}$ is an integer. Therefore, $\frac{by+r}{a}$ is an integer. Prop. I.

Hence the expression $\frac{by \pm r}{a}$ is always possible in positive whole numbers.

Cor. 2. Let c represent any constant quantity. Then will the expression $\frac{c \pm by}{a}$, under like circumstances, give different remainders. For whatever remainder results from the division of c by a, this also will be constant; and therefore whether we add to it or subtract from it, the unequal remainder, resulting from the division of by by a, the sums and the differences will likewise be unequal.

Hence, also, the expression $\frac{c \pm by}{a}$ is always possible in integers, either positive or negative, according to the relations between c and $b\,y$.

REMARKS.

1. If, in the last proposition, we give to y a value equal to a, or any multiple of a; the expression becomes $\frac{bma}{a}=bm$, leaving no remainder.

2. If we make $y=ma+1$, $ma+2$, &c., we shall have $\frac{bma+b}{a}$, $\frac{bma+2b}{a}$, &c., equal to $\frac{by}{a}$; from which we shall obviously obtain the same remainders as when we give to y the successive values, 1 . 2 . 3 . &c.

Whence it appears, that after y becomes a, or any multiple of a, the remainders will begin to return in the same order.

3. If a and b are not prime to each other, let m' be a common factor. Then we have $\frac{by}{a} = \frac{m'b'y}{m'a'} = \frac{m'}{m'} \times \frac{b'y}{a'}$, and the periods of the remainders will begin to return when $y = a'$, or ma', and will all be increased by the same multiplier m.

4. If in the expression $\frac{c \pm by}{a}$ we suppose a to be a composite number, as ma', the expression becomes $\frac{c \pm by}{ma'}$. Now when b is a multiple of either m or a', and y becomes a multiple of the other, the period of remainders, upon performing the division, will begin to return after a' terms, when b is a multiple of m, and after m terms, when it is a multiple of a'.

First, suppose that b is a multiple of m; then we have $\frac{c \pm by}{a} = \frac{c \pm mb'm'a'}{ma'} = \frac{c}{ma'} \pm b'm'$, or $\frac{c}{a} \pm b'm'$; from which will result the original remainder of c divided by a.

Again, suppose that b is a multiple of a'; then we have in the same way, $\frac{c \pm by}{a} = \frac{c \pm a'b'my'}{ma'} = \frac{c}{ma'} \pm b'y' =$ or $\frac{c}{a} \pm b'y'$, which gives the same remainder as before.

All the preceding propositions may be readily illustrated by the substitution of numbers.

[To be continued.]

ON THE COMPRESSIBILITY OF LIQUIDS.

By Pliny Earle Chase, Philadelphia, Pa.

Let $d =$ density, or specific gravity of a liquid (the specific gravity of air being 1).

If air is a perfectly elastic fluid, it would require, by the law of Mariotte, a pressure of d atmospheres to reduce it to the density d. By the same law, $\frac{1}{d+1} =$ additional compressibility of air at

density d, by 1 additional atmosphere; $d\left(\frac{1}{d} - \frac{1}{d+a}\right) = \frac{a}{d+a} =$ additional compressibility of air at density d, by a additional atmospheres.

It seems reasonable to suppose that the ratio of compressibility between the liquid and air of the same *density* may be some function of $d \times$ the compressibility of air under the same *pressure*. This function may perhaps be represented by the ratio of homologous representatives of equal weights of air and liquid under the ordinary atmospheric pressure; for example, by the ratio of the diameter of a sphere of air or liquid at density d, to the diameter of the same sphere at density 1, which is the ratio of $\frac{1}{d^{\frac{1}{3}}}$. If such is the case, $\frac{1}{d^{\frac{1}{3}}(d+1)} \times \frac{1}{2}$* $=$ compressibility of liquid by 1 additional atmosphere.

The assumption of the specific gravity of air as the unit of the ratio, is undoubtedly, to some extent, arbitrary. But as soon as the pressure of the air is removed, in any degree, from the surface of a liquid, a portion of the liquid assumes the form of vapor, and it would be impossible to approximate so nearly to the average density of the mixed liquid and vapor, at any other density of the air, as we can at the one we have assumed. It is true that even at the ordinary pressure of the atmosphere, liquids are constantly evaporating; but as that pressure is appointed by the Creator, as the one of natural equilibrium, it seems reasonable to presume that we shall not materially err, in taking it as the starting point of our hypothesis. To test that hypothesis, let us examine

Water,	which has $d = 812.5$	Mercury,	which has $d = 11023.7$
Alcohol,	" " $d = 642.7$	Sulphuric Ether,	" " $d = 581.7$

* $\frac{1}{2} =$ compressibility of air under pressure of a atmospheres, by the pressure of a additional atmospheres.

According to the above formula, the compressibility by a single additional atmosphere, would be, for

Water, $\frac{1}{\sqrt[3]{812.5 \times 813.5}} \times \frac{1}{2} = .00006586.$

Alcohol, $\frac{1}{\sqrt[3]{642.7 \times 643.7}} \times \frac{1}{2} = .00009001.$

Mercury, $\frac{1}{\sqrt[3]{11023.7 \times 11024.7}} \times \frac{1}{2} = .00000203.$

Sulph. Ether, $\frac{1}{\sqrt[3]{581.7 \times 582.7}} \times \frac{1}{2} = .00010279.$

In the following Table, the theoretical are compared with different experimental results.

Compressibility in Millionths, by 1 Atmosphere.	THEORY.	COLLADON and STURM.	DANIELL.	OERSTED.	CANTON.	PERKINS.
Of Water,	65.86	49 to 51	46.65	46	40 to 46	48 to 100
" Alcohol,	90.01	93 to 96	21.65	20	66	
" Mercury,	2.03	5	2.65	1	3	
" Sulph. Ether,	102.79	133 to 150	61.65	60		

The experimental results are undoubtedly all subject to correction for friction, imperfections of apparatus, and difficulty of minute observation; and the theoretical results should likewise be modified by considerations of temperature, vapor, imperfect elasticity of the air, and perhaps other unknown influences. Our data are so few and imperfect, that they cannot be regarded as deciding the truth or falsity of the theory. They may, however, be sufficient to call attention to an almost untrodden field of investigation, awakening an interest which will be followed by more numerous and accurate experiments, and lead to the modification of the formula here suggested, in the discovery of a new one, which will more accurately represent the law of fluid compressibility. It is already known that the law of Mariotte is only approximately true. When the compressibility of various different gases and vapors has been determined with the greatest possible

precision, we may, perhaps, theoretically determine, with equal precision, the compressibility of the condensed vapors in the liquid form.

The first experiments which proved with any tolerable degree of accuracy the compressibility of water were those of Perkins. The results of his early trials gave a degree of compressibility more than twice as great as his final estimate; but the ratios of compression between the successive experiments correspond very nearly, in many instances, with the theoretical ratios. I can hardly believe that this correspondence is entirely accidental.

In the following table, the first column gives the number of atmospheres (a) to which a column of water, 190 inches in length, was subjected; second column, the compressibility, in decimals of an inch, according to Perkins's table, with the ratios between the results of the successive experiments; third column, the theoretical compressibility in decimals of an inch, with the successive ratios. The numbers in the third column are obtained by the formula $190 \times \frac{a}{d^{\frac{1}{3}}(d+a)} \times \frac{1}{2}$.*

PRESSURE.	Compressibility according to Perkins.	Theoretical Compressibility.	PRESSURE.	Compressibility according to Perkins.	Theoretical Compressibility.
1 atmosph.	.019	.0125	150 atm's.	1.914	1.5866
	9.94	9.90		1.27	1.27
10 "	.189	.1238	200 "	2.440	2.011
	1.97	1.97		1.37	1.37
20 "	.372	.2438	300 "	3.339	2.7454
	1.47	1.48		1.26	1.22
30 "	.547	.3625	400 "	4.193	3.3587
	1.26	1.32		1.21	1.16
40 "	.691	.4777	500 "	5.087	3.8784
	1.17	1.24		1.16	1.11
50 "	.812	.5902	600 "	5.907	4.3247
	1.18	1.18		1.14	1.09
60 "	.956	.7001	700 "	6.715	4.7117
	1.10	1.15		1.10	1.07
70 "	1.056	.8075	800 "	7.402	5.051
	1.12	1.13		1.11	1.06
80 "	1.187	.9126	900 "	8.243	5.3505
	1.09	1.11		1.09	1.05
90 "	1.288	1.0152	1000 "	9.002	5.617
	1.10	1.10		1.76	1.29
100 "	1.422	1.1157	2000 "	15.833	7.2394
	1.35	1.42			

* 190 = length of experimental column; a = number of atmospheres; $d = 812.5$.

THE ELEMENTS OF QUATERNIONS.

By W. P. G. Bartlett, Cambridge, Mass.

The following brief essay is intended to present, in a direct method, the fundamental principles and notation of the *Quaternion Analysis*; so as to enable one who has mastered it to proceed at once, without difficulty, to the geometrical applications and further developments given in the VIIth of Hamilton's "*Lectures on Quaternions*" or elsewhere. With a few trifling exceptions, Hamilton's notation is strictly retained. Nothing will be given, at present, upon the *differentials of quaternions*, which are introduced in Lecture VII., § xcvii., of Hamilton.

The author intends hereafter to take up, as a continuation of these papers, some special subject, such as Spherical Trigonometry, or Analytic Geometry, and discuss it in the language of Quaternions.

I. — Lines.

1. *Equal* lines are such as have the same length and the same direction in space.

2. One line is the *negative* of another when it has the same length, but the opposite direction.

3. The *sum* of two lines is the diagonal of a parallelogram, of which these lines are two adjacent sides; that diagonal being taken, whose direction lies between the directions of the two given lines. Hence the sum of two or more lines is the same, in whatever order they may be placed or added together; and when any set of lines in space form a closed circuit, their sum is equal to zero, and therefore each line is the negative of the sum of all the others.

4. In operations on lines in space, it is convenient to substitute for some of the given lines other lines, equal, and therefore parallel, to the given ones, but passing through a point in space com-

mon to themselves and the other given lines; so that all the lines may be coöriginal. In what follows this is supposed to be the case.

II. — Quaternions.

5. The abstract operation of so changing the length and direction of one line as to make it coincide in length and direction with another line, is considered as the *quotient* of the second line *divided* by the first, and is called a *quaternion.* The plane in which, by § 4, both these lines are situated, is called the *plane* of the quaternion. Conversely the *product* of *multiplying* the first line by the quaternion is the second line. In writing a product the *multiplier* always precedes the *multiplicand.*

6. Lines are generally denoted by the letters α, β, γ, &c.; quaternions by p, q, r, &c. The angular distance in the plane of α and β from the positive direction of α to that of β is denoted by $\frac{\beta}{\alpha}$.

7. The quotients $\beta \div \alpha$, and $\delta \div \gamma$, are *equal,* when α, β, γ, and δ being coplanar, $\frac{\beta}{\alpha} = \frac{\delta}{\gamma}$, and the ratio of the length of β to that of α is equal to ratio of the length of δ to that of γ. Hence if there are given any two quaternions, $q = \beta' \div \alpha'$, and $p = \gamma' \div \delta'$, there can always be found three other lines, α, β, and γ, such that $\beta \div \alpha = \beta' \div \alpha' = q$, or $\beta = q\,\alpha$; and $\gamma \div \alpha = \gamma' \div \delta' = p$, or $\gamma = p\,\alpha$; that is, some line, α, can always be found such that it may be multiplied by both the given quaternions. The three lines α, β, and γ might have been so determined that $\beta \div \alpha = \beta' \div \alpha' = q$, and $\gamma \div \beta = \gamma' \div \delta' = p$.

8. The *sum* of two quaternions is defined by the equations

$$p = \beta \div \alpha,\ q = \gamma \div \alpha,\ p + q = \beta \div \alpha + \gamma \div \alpha = (\beta + \gamma) \div \alpha;$$

or, $$p\,\alpha = \beta,\ q\,\alpha = \gamma,\ (p + q)\,\alpha = \beta + \gamma = p\,\alpha + q\,\alpha.$$

By §§ 3 and 5, $p + q = (\beta + \gamma) \div \alpha$ must be a quaternion. Hence

$\Sigma q = q'$; that is, the sum of two or more quaternions is a quaternion; and, by § 3, it is the same in whatever order its component quaternions may be taken.

9. The *product* of multiplying q by p is defined by the equations

$$q = \beta \div \alpha,\ p = \gamma \div \beta,\ p\,q = \gamma \div \alpha\ ;$$

or, $$q\,\alpha = \beta,\ p\,\beta = \gamma,\ p\,q\,.\,\alpha = p\,.\,q\,\alpha^{*} = p\,\beta = \gamma.$$

From its form it follows, as in § 8, that $\Pi\, q = q'$; that is, the product of two or more quaternions is a quaternion. If $p\,q = r$, we may write $p = r \div q$, which defines the *quotient* of two quaternions. But we cannot write $q = r \div p$, unless $q\,p = r$; and it must be observed that a quaternion is not, in general, *commutative;* that is, independent of the order of its factors; for, let the additional lines δ and ε be so determined that $\delta \div \beta = \beta \div \alpha = q$, and $\beta \div \varepsilon = \gamma \div \beta = p$; whence $q\,p = \delta \div \varepsilon$; but by § 7, δ must be in the plane of α and β, and ε in that of β and γ; therefore δ and ε cannot be in the plane of α and γ, unless β is also in this plane; so that $q\,p$ is different from $p\,q$, unless p and q are *coplanar.*

(To be continued.)

REPORT OF THE JUDGES ON PRIZE ESSAYS.

Only three essays were presented for our consideration; one on Central Forces; one on Spherical Conics; and one on Projectiles. We are of opinion that the first alone has sufficient merit to entitle it to a prize; and to this, as the "best essay," we award the prize of Fifty Dollars. Prof. Perkins was unable to concur in this report.

William Ferrel.
John B. Henck.

September 10*th*, 1859.

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* Points are used, in this case, to separate a product into its proper factors; thus, this equation may be read $p\,q$ times $\alpha = p$ times $q\,\alpha$.

Editorial Items.

The following gentlemen have sent us solutions of the Prize Problems in the July number of the Monthly: — Gustavus Frankenstein, Springfield, Ohio; F. T. Hampton, Columbian College, Washington, D. C.; Roland Thompson, Junior Class, Jefferson College, Canonsburg, Penn.; Otho E. Michaelis, Freshman Class, Free Academy, N. Y.; David Trowbridge, Perry City, Schuyler Co., N. Y.; J. T. Lovegrove, Baltimore, Md.

It gives us pleasure to announce that the well-known and enterprising firm of Ivison & Phinney have become the publishers of the Mathematical Monthly. We are satisfied that their energy and large business facilities not only guarantees the permanency of the Monthly, but a wider usefulness and a larger measure of success. In advance of a fuller announcement, Messrs. Ivison & Phinney authorize us to say, that the continued issue of the Mathematical Monthly is now placed beyond any contingency; that it will continue to be manufactured in Cambridge, under the supervision of the Editor; that its present character of being among the finest specimens of mathematical printing ever executed, will be fully maintained; that the size of the second and succeeding volumes will not fall below Vol. I.; that each volume will contain at least two steel plates, not inferior in artistic excellence to those illustrating Donati's Comet, in Vol. I.; and lastly, that the Monthly will be enlarged and improved just as fast as its circulation will warrant. They propose to issue a largely increased edition, and bring it to the attention of all the prominent professors, teachers, engineers, &c., in the country at all likely to be interested in such a publication. The Editor would improve the opportunity to say, for the benefit of those who may never have seen the first number of Vol. I., in which he has developed pretty fully the plan and aims of the Monthly, that its primary aim is educational, and it is designed to meet the wants of teachers and students generally. At the same time it will embrace in its pages investigations in the higher mathematics, and thus secure the coöperation of those who are enlarging the boundaries of this science. We earnestly invite the attention of all to the claims of the Monthly, and hope that, during the coming year, the number of those who are now coöperating with us to secure the ends of its establishment, may be largely increased. All contributions should still be addressed to the Editor, whose editorial relations to the Monthly are entirely unchanged.

Our Prizes. The success of these prizes during the past year, fully warrants their continuance, with the following modifications: — In each number Five Prize Problems will, as heretofore, be published, of which the first and second will be quite elementary; and for the best solution of which a prize of a copy of the Monthly is offered, by Messrs. Ivison & Phinney. This prize is open to all students in Academies, High Schools, and all institutions, whether public or private, not conferring degrees. The successful competitor will receive the volume free. The remaining three of the five problems will be open to the same competition as heretofore, with the following prizes: For the best solutions we offer a first prize of six dollars, and a second prize of four dollars to those of the second order of merit. The prize solutions will continue to be published. Judges: Simon Newcomb, Esq., W. P. G. Bartlett, Esq., and Truman Henry Safford, Esq. . . . . The prizes for essays remain the same, and are offered under the same conditions, except that the first and second are open to all competitors; and it shall be discretionary with the judges to decide to which of the prizes the best essay, the second in order of merit, &c., are entitled. Judges: William Ferrel, Esq., J. B. Henck, Esq., and Chauncey Wright, Esq. Mr. Wright takes the place of Prof. Perkins, who declines to serve another year. See pp. xi. and xii., No. for October, Vol. I., for conditions.

THE

MATHEMATICAL MONTHLY.

Vol. II... NOVEMBER, 1859. No. II.

PRIZE PROBLEMS FOR STUDENTS.

I. Form the equation, the roots of which are $a+\sqrt{b}, a-\sqrt{b}, -c$ and d

II. Find the equated time of payment of two sums S and s, due respectively at the end of T and t years, allowing simple interest.

III. Of all right-angled plane triangles having the same given hypothenuse, to find the one whose area is the greatest possible. To be solved by Algebra.

IV. What is that fraction, the cube of which being subtracted from it, the remainder is the greatest possible? To be solved by Algebra.

V. If, in a plane or spherical triangle, A, B, C denote the angles, and a, b, c the sides respectively opposite them; and if we produce the sides of the triangle, and consider the three circles which touch two of the sides interiorly and the third side exteriorly; and denote by r, ϱ the radii of the circumscribed and inscribed circles; by $\varrho', \varrho'', \varrho'''$, the radii of the circles touching exteriorly the sides a, b, c respectively; by $\delta', \delta'', \delta'''$, the distances of the centres of

these circles from the centre of the circle circumscribed about the primitive triangle; then we have in the plane

$$-\sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2} A \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C,$$
$$\sin A - \sin B + \sin C = 4 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} B \sin \tfrac{1}{2} C,$$
$$\sin A + \sin B - \sin C = 4 \sin \tfrac{1}{2} A \sin \tfrac{1}{2} B \cos \tfrac{1}{2} C,$$

and in the sphere

$$-\sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2} A \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C \,.\, \frac{\cos \delta'}{\cos r \cos \varrho'},$$
$$\sin A - \sin B + \sin C = 4 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} B \sin \tfrac{1}{2} C \,.\, \frac{\cos \delta''}{\cos r \cos \varrho''},$$
$$\sin A + \sin B - \sin C = 4 \sin \tfrac{1}{2} A \sin \tfrac{1}{2} B \cos \tfrac{1}{2} C \,.\, \frac{\cos \delta'''}{\cos r \cos \varrho'''}.$$

The solution of these problems must be received by the first of January, 1860. Problem V. is one of the analogies by Prof. CHAUVENET, to which we referred in the last number of the MONTHTY.

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. X., Vol. I.

THE first Prize is awarded to GUSTAVUS FRANKENSTEIN, of Springfield, Ohio.

The second Prize is awarded to ROLAND THOMPSON, Jefferson College, Canonsburg, Pa.

PRIZE SOLUTION OF PROBLEM II.

BY ALL THE COMPETITORS.

"If A, B, C be the angles, and a, b, c the opposite sides, in a plane triangle, of which S denotes the surface; prove that

$$a^2 + b^2 + c^2 = 4\, S\,(\cot A + \cot B + \cot C)."$$

By Trigonometry,

$$a^2 = b^2 + c^2 - 2\, b\, c \cos A$$

$$b^2 = a^2 + c^2 - 2\,a\,c \cos B$$
$$c^2 = a^2 + b^2 - 2\,a\,b \cos C.$$

By addition and transposition we get

$$(1)\quad a^2 + b^2 + c^2 = 2\,(b\,c \cos A + a\,c \cos B + a\,b \cos C).$$

But $2\,S = b\,c \sin A = a\,c \sin B = a\,b \sin C$:

or $$b\,c = \frac{2\,S}{\sin A},\quad a\,c = \frac{2\,S}{\sin B},\quad a\,b = \frac{2\,S}{\sin C}.$$

Substituting these values of $b\,c$, $a\,c$, $a\,b$, in (1), we obtain

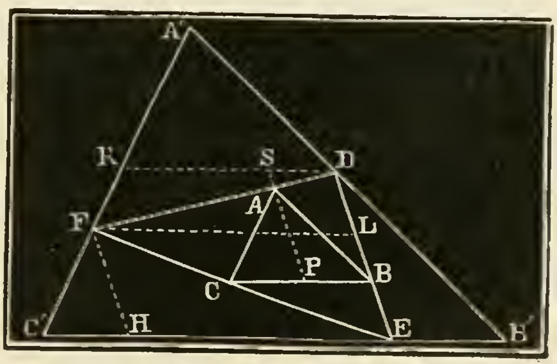
$$a^2 + b^2 + c^2 = 4\,S(\cot A + \cot B \cot C).$$

Prize Solution of Problem III.

BY GUSTAVUS FRANKENSTEIN, SPRINGFIELD, OHIO.

"If one of the similar triangles $A\,B\,C$ and $A'\,B'\,C'$ be inscribed in the triangle $D\,E\,F$ and the other circumscribed about it; prove that the area of $D\,E\,F$ will be a mean proportional between the areas of $A\,B\,C$ and $A'\,B'\,C'$."

Let DEF be any triangle, $A\,B\,C$ an inscribed, and $A'B'\,C'$ a similar circumscribed, triangle. Draw $D\,S\,R$ and FL parallel to $B\,C$ or $B'\,C'$; also, FH and SAP parallel to DE. By reason of these parallels, the triangle $B'ED$ is similar to $B\,P\,A$, $EB\,C$ to ELF, FLD to DSA, and $C'HF$ to CPA; whence the proportions:



(1) . . . $B'E : ED = BP : PA$ (2) . . . . $EB : B\,C = EL : LF$

(3) . . . . $LF : DL = SD : SA$ (4) . . . . $C'H : HF = PC : PA$;

in which

$$FH = LE = DE - DL,\quad SD = PB,\quad BC = BP + PC,$$
$$HE = LF,\quad SA + AP + BE = DE.$$

But (2) and (4) give

$$(DE - DL)(BP + PC) = EB \times LF,\ (DE - DL)\,PC = C'H \times PA;$$

hence $\quad (BP + PC) \times C'H \times PA = PC \times EB \times LF,$

or, $\quad PC \times (EB \times FL - C'H \times PA) = BP \times C'H \times PA;$

and as $$PC \times (DE - DL) = C'H \times PA,$$

therefore $\quad (DE - DL)\, BP = (EB \times LF - C'H \times PA).$

Substitute for DE its value $\frac{B'E \times PA}{BP}$ deduced from (1), and for DL its value $\frac{FL \times SA}{BP}$ deduced from (3), and there results the condition,

$$(5) \quad B'E \times PA - FL \times SA = EB \times LF - C'H \times PA,$$

or $$(B'E + C'H)\, PA = (EB + SA)\, FL.$$

Add $EH \times PA$ to the first side of this equation, and its equal $FL \times PA$ to the second; then, because

$$B'E + EH + C'H = B'C', \; SA + AP + BE = DE,$$

(5) becomes

$$(6) \qquad B'C' \times PA = DE \times LF.$$

But in the triangles ABC and DEF, the angle $APB = FLE$;

$$(7) \; \therefore \quad ABC : DEF = AP \times BC : FL \times DE;$$

or, denoting the area of the triangle ABC by t, and by (6),

$$t : DEF = AP \times BC : B'C' \times AP,$$

$$\therefore DEF = t \times \frac{B'C' \times AP}{BC \times AP} = t \times \frac{B'C'}{BC};$$

Also, $$ABC : A'B'C' = BC^2 : B'C'^2;$$

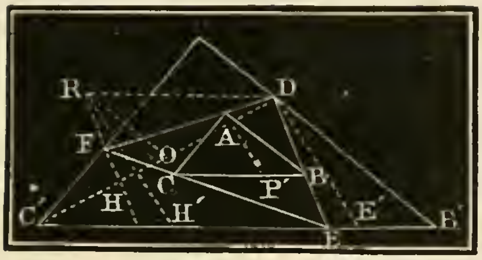
or $$A'B'C' = t \times \frac{B'C'^2}{BC^2}$$

$$\therefore ABC \times A'B'C' = t^2 \times \frac{B'C'^2}{BC^2} = (DEF)^2.$$

Therefore DEF is a mean proportional between ABC and $A'B'C'$.

Corollary. The property $B'C' \times PA = DE \times (LF \text{ or } EH)$ is not confined to the side DE of the triangle DEF, and its par-

allel AP, but applies to any line DE' and its parallel AP'. For



the triangle DEF is half of the parallelogram DH; having the same base and altitude; and for the same reason the parallelogram $DH = DH'$. Hence, as the angle $AP'C = H'E'D$,

$$\therefore \quad t : DEF = BC \times AP' : (HE \text{ or } H'E)\, DE',$$

$$\therefore \quad DEF = t \times \frac{HE \times DE'}{BC \times AP'},$$

$$\therefore \quad (DEF)^2 = t^2 \times \frac{HE^2 \times DE'^2}{BC^2 \times AP'^2} = t^2 \times \frac{B'C'^2}{BC^2},$$

$$\therefore \quad HE \times DE' = B'C' \times AP'.$$

Therefore the line DE' may take the direction DB', and we shall have

$$B'C' \times AB = DB' \times EH;$$

so that drawing $C'D$, and RO parallel to DB', intersecting $C'D$ in O, we must have $RO = AB$. For the triangles $C'DB'$ and DRO are similar, having their sides respectively parallel, giving

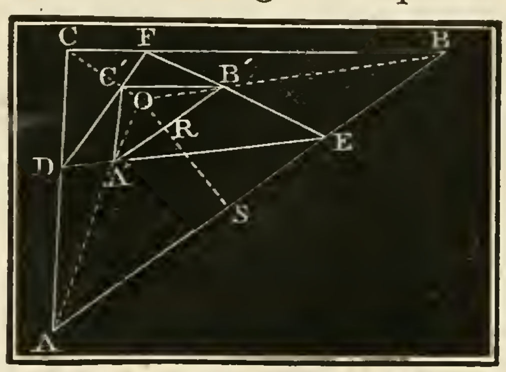
$$B'C' : B'D = RD \text{ or } EH : RO.$$

$$\therefore \; DB' \times EH = B'C' \times RO = B'C' \times AB. \quad \therefore \; RO = AB.$$

Second Solution of Prize Problem III.

BY ROLAND THOMPSON, JEFFERSON COLLEGE, CANONSBURG, PA.

Inscribe the triangle $A'B'C'$ in DEF, and through the points D, E, F, draw straight lines respectively parallel to the sides of $A'B'C'$; the triangle ABC thus formed is similar to $A'B'C'$. The three straight lines passing through the homologous angles will meet in the same point. For the



straight lines CC' and BB' produced will meet in some point

O, and if $A\,O$ does not pass through A', let A'' be the point in which it meets the line $A'B'$. Then

$$B\,O : B'\,O = A\,B : A''\,B'$$
$$B\,O : B'\,O = B\,C : B'C';$$
$$\therefore \qquad A\,B : A''\,B' = B\,C : B'\,C'.$$

Therefore the triangle $A''\,B'\,C'$ is similar to $A\,B\,C$ and equal to $A'\,B'\,C'$, and therefore $A''\,B' = A'\,B'$, and $A\,O$ passes through A'. Let $O\,S$ be drawn perpendicular to $A\,B$, and $A'\,B'$; then, since $O\,S$, $O\,R$, and $R\,S$, are the altitudes of the triangles $A\,O\,B$, $A'\,O\,B'$, and $A'\,B'\,E$, we have

$$(1)\quad \frac{A\,O\,B}{A'\,O\,B'\,E} = \frac{A\,O\,B}{A'\,O\,B' + A'\,B'\,E} = \frac{A\,B \times O\,S}{A'\,B' \times O\,R + A'\,B' \times R\,S}$$
$$= \frac{A\,B \times O\,S}{A'\,B' \times O\,S} = \frac{A\,B}{A'\,B'} = \frac{O\,S}{O\,R} = \frac{A'\,O\,B'\,E}{A'\,O\,B'}.$$

In like manner

$$(2)\qquad \frac{B\,O\,C}{B'\,F\,C'\,O} = \frac{B'\,F\,C'\,O}{B'\,O\,C'} = \frac{B\,C}{B'\,C'} = \frac{A\,B}{A'\,B'}.$$

$$(3)\qquad \frac{A\,O\,C}{A'\,O\,C'\,D} = \frac{A'\,O\,C'\,D}{A'\,O\,C'} = \frac{A\,C}{A'\,C'} = \frac{A\,B}{A'\,B'}.$$

Since (1), (2), and (3), are equal ratios, we get, by addition,

$$\frac{A\,B\,C}{D\,E\,F} = \frac{D\,E\,F}{A'\,B'\,C'},$$

showing that $D\,E\,F$ is a mean proportional between $A\,B\,C$ and $A'\,B'\,C'$ as was to be proved.

Prize Solution of Problem IV.

BY ALL THE COMPETITORS.

"If a be one of the sides of an equilateral spherical triangle and A one of its angles, prove that $\sec A = \sec a + 1$."

If $a = b = c$ in the fundamental equation

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

it becomes

$$\cos a = \cos^2 a + \sin^2 a \cos A.$$

$$\therefore \quad \cos A = \frac{\cos a - \cos^2 a}{\sin^2 a} = \frac{\cos a\,(1 - \cos a)}{(1 + \cos a)(1 - \cos a)} = \frac{\cos a}{1 + \cos a}.$$

$$\therefore \quad \sec A = \frac{1 + \cos a}{\cos a} = \sec a + 1.$$

Prize Solution of Problem V.

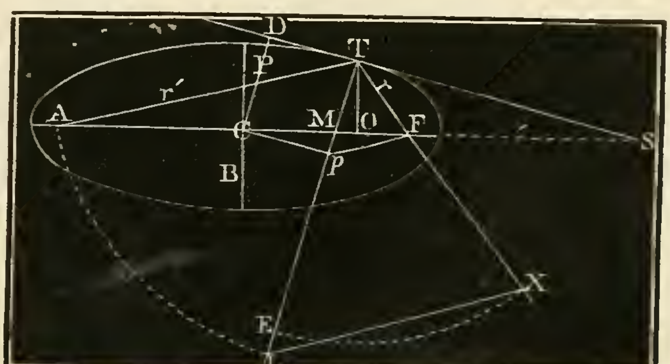
BY GUSTAVUS FRANKENSTEIN, SPRINGFIELD, OHIO.

"If the semiaxes of an ellipse be A and B, P the length of the perpendicular dropped from the centre on the tangent to the curve, r and r' the distances from the point of tangency to the foci, and ϱ the radius of curvature at this point; prove that

$$\varrho = \frac{A^2B^2}{P^3} = \frac{rr'}{P};$$

and from this theorem construct the corresponding point of the evolute."

1. The equation of the ellipse referred to its centre and axes is $A^2x^2 + B^2y^2 = A^2B^2$; and if x', y' denote the coördinates of the point of tangency, the equation of the tangent line is $A^2yy' + B^2xx' = A^2B^2$. The tangent of the angle which the line DS makes with the axis of x is $D_xy = -\frac{B^2x'}{A^2y'}$; or if we take the angle TSC, then $\tan TSC = \tan MTO = \frac{B^2x'}{A^2y'} = \alpha$.



If in the equation of the tangent line $y = 0$, then $CS = x = \frac{A^2}{x'}$; the normal $MT = N = y'\sqrt{1 + \alpha^2}$; subnormal $MO = \alpha y'$; the subtangent $SO = \frac{y'}{\alpha}$, the similar triangles TSM and DSC give $DC : CS = TM : MO + OS$;

or

$$P : \frac{A^2}{x'} = N : \alpha y' + \frac{y'}{\alpha}.$$

$$\therefore \qquad P = \frac{A^2 N}{x'} \times \frac{\alpha}{y'(1+\alpha^2)} = \frac{A^2 N}{x'} \times \frac{\alpha y'}{N^2} = \frac{B^2}{N}.$$

But from the Calculus we have the radius of curvature, $\varrho = \frac{A^2 N^3}{B^4}$;

$$\therefore \qquad \varrho \times P^3 = \frac{A^2 N^3}{B^4} \times \frac{B^6}{N^3} = A^2 B^2. \quad \therefore \varrho = \frac{A^2 B^2}{P^3}.$$

Further, the normal, MT, bisects the angle ATF;

$$\therefore \qquad \frac{r'}{r} = \frac{AM}{MF}, \frac{r'+r}{r} = \frac{2A}{r} = \frac{AM+MF}{MF} = \frac{2\sqrt{A^2-B^2}}{MF}.$$

$$\therefore \qquad MF = \frac{r\sqrt{A^2-B^2}}{A}, \text{ and } AM = \frac{r'\sqrt{A^2-B^2}}{A}.$$

Also, in the triangle ATF we have

$$AT \times TF = AM \times MF + MT^2;$$

or,
$$r'r = \frac{r'r(A^2-B^2)}{A^2} + N^2. \quad \therefore r'r = \frac{A^2 N^2}{B^2}.$$

Dividing by $P = \frac{B^2}{N}$ gives $\frac{r'r}{P} = \frac{A^2N^3}{B^4} = \varrho$.

2. To find by construction the corresponding point in the evolute. We have just found that $r' : P = \varrho : r$. Produce the normal TM, and also TF. On TM produced take $Tp = P$, and $TL = r'$. Draw pF, and LX parallel to pF; then

$$TL : Tp = TX : TF;$$

or
$$r' : P = TX : r'. \quad \therefore TX = TE = \varrho.$$

This construction is by Frederic T. Hampton.

Chauncey Wright.
Truman Henry Safford.

SOLUTION OF PROBLEMS IN MAXIMA AND MINIMA BY ALGEBRA.

By Ramchundra, Late Teacher of Science, Delhi College.

1. *To divide a given number into two such parts that their product may be the greatest possible.*

Put the given number $= a$, one of the parts required $= x$, and consequently $a - x =$ the other part. Therefore $x(a - x) = ax - x^2 =$ product $=$ maximum $= r$. Therefore $x^2 - ax = -r$. Solving this quadratic equation we find

$$x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - r}.$$

Now it is evident that r cannot be greater than $\frac{1}{4}a^2$; for if it be so, the value of x becomes impossible; therefore the product $ax - x^2$, or r, is greatest when $\frac{1}{4}a^2 = r$; therefore $x = \frac{a}{2}$.

Second Solution without Impossible Roots.

In the expression $ax - x^2$, which is to become a maximum, let $x = y + \frac{a}{2}$, where the value of y, determined by the condition of $ax - x^2$ being a maximum, will show whether it is positive, zero, or negative. We now find

$$ax - x^2 = ay + \frac{a^2}{2} - y^2 - ay - \frac{a^2}{4} = \frac{a^2}{4} - y^2,$$

which is evidently a maximum when $y = 0$. Therefore $x = \frac{a}{2}$ as before.

II. *Of all right-angled plane triangles having the same hypothenuse, to find that whose area is the greatest possible.*

Let $a =$ hypothenuse, $x =$ base, $y =$ perpendicular. Then, $x^2 + y^2 = a^2$, we shall have $y = \sqrt{a^2 - x^2}$, and consequently $\frac{xy}{2} = \frac{x}{2}\sqrt{a^2 - x^2} =$ the area of the triangle $=$ maximum, and consequently the square of the area, or $\frac{1}{4}(a^2x^2 - x^4) =$ max., and also

four times this, or $a^2x^2 - x^4 = \max. = r$. Therefore $x^4 - a^2x^2 = -r$. Solving this quadratic equation, we find

$$x^2 = \frac{a^2}{2} \pm \sqrt{\frac{a^4}{4} - r},$$

and it is manifest that $a^2x^2 - x^4$, or r cannot be greater than $\frac{1}{4}a^4$; therefore when $r =$ maximum, we must have $r = \frac{1}{4}a^4$, therefore $x^2 = \frac{1}{2}a^2$, and $x = \frac{a}{\sqrt{2}}$, and $y = \sqrt{a^2 - x^2} = \frac{a}{\sqrt{2}}$. Hence it appears that the right-angled plane triangle contains the greatest area whose two sides, containing the right angle, are equal to each other.

Second Solution without Impossible Roots.

In the expression $a^2x^2 - x^4$, which is to become a maximum, let $x^2 = y^2 + \frac{a^2}{2}$.

$$\therefore \quad a^2x^2 - x^4 = a^2\left(y^2 + \frac{a^2}{2}\right) - \left(y^2 + \frac{a^2}{2}\right)^2$$

$$= a^2y^2 + \frac{a^4}{2} - y^4 - a^2y^2 - \frac{a^4}{4} = \frac{a^4}{4} - y^4.$$

which is evidently a maximum, when $y^4 = 0$, and therefore $x^2 = \frac{a^2}{2}$, or $x = \frac{a}{\sqrt{2}}$, as before.

III. *To bisect a triangle by the shortest line.*

Let ABC be the given triangle, and DQ the shortest line required. Also, let $CD = x$, $CQ = y$, $DQ = u$, and a, b, c the three sides of the triangle, and C the angle BCA. DM and BN are perpendiculars, drawn from the points D and B, on the line CA. Now, by similar triangles, we find $\frac{DM}{CD} = \frac{BN}{CB} = \sin C$; therefore $DM = x \sin C$, and $BN = a \sin C$, and

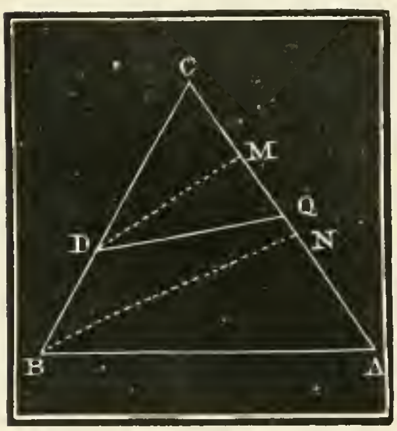


Fig.

$$\therefore \quad \frac{CQ \times DM}{2} = \frac{xy \sin C}{2}; \quad \frac{CA \times BN}{2} = \frac{ab \sin C}{2};$$

but, by supposition,

$$2 \times \frac{xy \sin C}{2} = \frac{ab \sin C}{2}; \ \therefore \ ab = 2xy; \ \therefore \ y = \frac{a\,b}{2\,x}.$$

By Proposition 13, Book 2d, of EUCLID, we find

$$u^2 = x^2 + y^2 - 2\,x\,y \cos C = x^2 + \frac{a^2 b^2}{4\,x^2} - a\,b \cos C = \text{min.} = r.$$

$$\therefore \qquad x^4 + \frac{a^2 b^2}{4} - a\,b\,x^2 \cos C = r\,x^2;$$

$$\therefore \qquad x^4 - (a\,b \cos C + r)\,x^2 = -\frac{a^2 b^2}{4}.$$

Completing the square, and extracting the square root, we find,

$$x^2 = \frac{a\,b \cos c + r}{2} \pm \sqrt{\frac{(a\,b \cos C + r)^2 - a^2 b^2}{4}}.$$

Now $a^2 b^2$ is greater than $a^2 b^2 \cos^2 C$; therefore in order that the value of x^2 may not become impossible, we must have $a\,b \cos C + r = a\,b$; therefore $r = a\,b - a\,b \cos C$, and therefore, when $r =$ min., we must have

$$x^2 = \frac{a\,b \cos C + r}{2} = \frac{a\,b}{2}; \ \therefore x = \sqrt{\frac{a\,b}{2}}, \, y = \frac{a\,b}{2\,x} = \sqrt{\frac{a\,b}{2}}.$$

$$\therefore \qquad u^2 = \frac{a\,b}{2} + \frac{a\,b}{2} - a\,b \cos C = a\,b\,(1 - \cos C)$$

$$= a\,b\left(\frac{2\,a\,b + c^2 - (a^2 + b^2)}{2\,a\,b}\right) = \frac{c^2 - (a - b)^2}{2};$$

and therefore, $u = \sqrt{\frac{(c - a + b)\,(c + a - b)}{2}}$.

SECOND SOLUTION WITHOUT IMPOSSIBLE ROOTS.

Let $a\,b \cos C + r = A$, and $\frac{a^2 b^2}{4} = B$; therefore the equation $x^4 - (a\,b \cos C + r)\,x^2 = -\frac{a^2 b^2}{4}$ becomes $x^4 - A\,x^2 = -B$.

Also, let $x^2 = y + \frac{A}{2}$,

$$\therefore \qquad x^4 - A\,x^2 = y^2 + A\,y + \frac{A^2}{4} - A\,y - \frac{A^2}{2}$$

$$= y^2 - \frac{A^2}{4} = -B. \quad \therefore \ y^2 + B = \frac{A^2}{4}.$$

$\therefore$ When $\frac{A^2}{4}$ or $r = \text{min.}$, $y = 0$, $\therefore B = \frac{A^2}{4}$, or $\frac{a^2 b^2}{4} = \frac{A^2}{4}$, and $a\,b = A = a\,b \cos C + r$, and

$$r = a\,b - a\,b \cos C = a\,b\,(1 - \cos C),$$

and $x^2 = \frac{A}{2} = \frac{a\,b \cos C + r}{2} = \frac{a\,b}{2}$, $\therefore x = \sqrt{\frac{a\,b}{2}}$, as before.

IV. *To find the least parabola which shall circumscribe a given circle.*

Since the parabola and the circle touch at E, therefore PE is a normal to the parabola, and PB is the subnormal = semi-parameter. Let $PB = z$; therefore the equation of the parabola is $y^2 = 2\,z\,x$. Also,

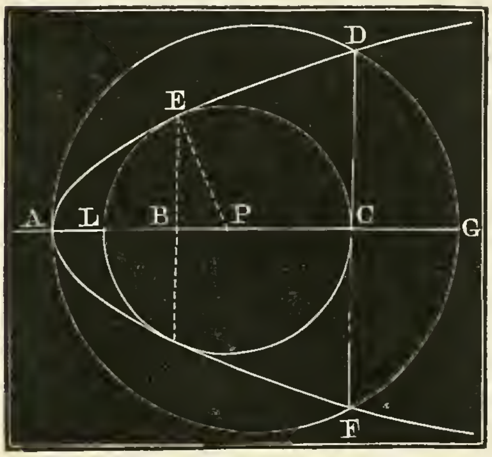


Fig.

$$EB^2 = r^2 - z^2, \ \therefore AB = \frac{r^2 - z^2}{2\,z}.$$

$$A\,C = AB + B\,P + P\,C$$

$$= \frac{r^2 - z^2}{2\,z} + z + r = \frac{(r+z)^2}{2\,z}.$$

Now the area of the parabola $FAD = \frac{4}{3}\,A\,C \times CF$ and $CF = \sqrt{2\,z\,.\,A\,C}$.

$$\therefore \text{ Area } FAD = \tfrac{4}{3}\,A\,C \times \sqrt{2\,z\,.\,A\,C} = \frac{2\,(r+z)^3}{3\,z} = u = \text{min.}$$

Let $r + z = y$; therefore $z = y - r$. Therefore $\frac{y^3}{y-r} = \text{min.} = u$, and therefore $y^3 - u\,y + u\,r = 0$. Let one of the negative roots of this equation $= -a$, and therefore $y + a$ must exactly divide the equation $y^3 - u\,y + u\,r = 0$.

$$
\begin{array}{r l}
y + a) & y^3 - u\,y + u\,r = 0 \ \big(y^2 - a\,y + a^2 - u = 0 \ldots (B) \\
& \underline{y^3 + a\,y^2} \\
& \quad - a\,y^2 - u\,y \\
& \quad \underline{- a\,y^2 - a^2\,y} \\
& \qquad (a^2 - u)\,y + u\,r \\
& \qquad \underline{(a^2 - u)\,y + a^3 - a\,u} \\
& \qquad\qquad u\,r - a^3 + a\,u = 0\,;
\end{array}
$$

and therefore $a^3 - au = ru$; $\therefore u = \frac{a^3}{a+r}$.

Now solving the quadratic (B), we find

$$y = \frac{a}{2} \pm \sqrt{u - \frac{3a^2}{4}};$$

and in order that u may become a minimum, that is, the least value of u which will make y real, we must have

$$u = \frac{3a^2}{4}, \therefore \frac{a^3}{a+r} = \frac{3a^2}{4}, \text{ or } \frac{a}{a+r} = \tfrac{3}{4}, \therefore 3a + 3r = 4a;$$

$$\therefore \quad a = 3r, \therefore y = \frac{a}{2} = \frac{3r}{2}. \quad \therefore z = y - r = \frac{3r}{2} - r = \frac{r}{2}.$$

Second Solution without Impossible Roots.

In the equation $y^2 - ay + a^2 - u = 0$, or $y^2 - ay = u - a^2$, let $y = w + \frac{a}{2}$, and

$$\therefore \quad y^2 - ay = w^2 + aw + \frac{a^2}{4} - aw - \frac{a^2}{2} = w^2 - \frac{a^2}{4} = u - a^2.$$

$\therefore u = w^2 + \frac{3a^2}{4}$; which is evidently a minimum, when $w^2 = 0$.

$\therefore u = \frac{3a^2}{4}$; but $u = \frac{a^3}{a+r} = \frac{3a^2}{4}$, or $4a = 3a + 3r$, and $a = 3r$, and therefore $y = \frac{a}{2} = \frac{3r}{2}$ and $z = y - r = \frac{r}{2}$, as before.

The foregoing problems, extracted from a Treatise on Problems of Maxima and Minima solved by Algebra, must suffice as examples of the author's method. (See Notice.)

NOTE ON THE FORTY-SEVENTH PROPOSITION OF EUCLID.

BY JOHN M. RICHARDSON,
Professor of Mathematics, Collegiate Institute, Bowdon, Ga.

The square described on the hypothenuse of a right-triangle is equal in area to the sum of the squares on the other two sides.

HISTORICAL ITEMS.

On account of its extensive application, this is one of the most interesting propositions in elementary plane geometry. It is variously denominated the Pythagorean Proposition, the Hecatomb Proposition, the Carpenter's Theorem, the Pons Asinorum of Mathematics. The name of its reputed discoverer, PYTHAGORUS, has given rise to the first; the sacrifice of a hundred fat oxen, which he made as a thank offering to the gods for the mental illumination that enabled him to discover it, to the second; its frequent application in building, to the third; and its supposed difficulty, to the fourth.

There seems to be some dispute, however, as to whether the fourth of the above names was given to the Vth, the XXth, or the XLVIIth proposition of EUCLID.

"PROCLUS in his 'Commentaries on EUCLID,'" says, "that the Epicureans derided the XXth proposition as being manifest 'even to asses;' for if a bundle of hay were placed at one extremity of the base of a triangle, and an ass at the other, the animal would not be *such an ass* as to take the crooked path to the hay instead of the straight one; as he would know the direct course to be the shorter: *this* was therefore called the 'asses' bridge.'"

CAMERER, in his notes on the first six books of EUCLID, collected *seventeen* demonstrations of this theorem.

HOFFMANN, in 1819, published a collection of *thirty-three* different solutions of it. I send you *twenty-eight* as the result of a short investigation. I have had no opportunity of comparing them with the other collections mentioned, but have given credit whenever I could; and with regard to the others, although they have resulted from my own hurried examinations, I cannot suppose that any of them are new. They have, doubtless, been discovered by different investigators over and over again. Many, however, may be new to the readers of the MONTHLY, and I send them as exercises for students.

Demonstration 1. Fig. 1. EUCLID. Triangle $ABG = \frac{1}{2}ACGF = ACM = \frac{1}{2}AKML$.

$\therefore ACGF = AKML.$

Also, $BCDE = BKLH$.

$\therefore ACGF + BCDE = ABMH$

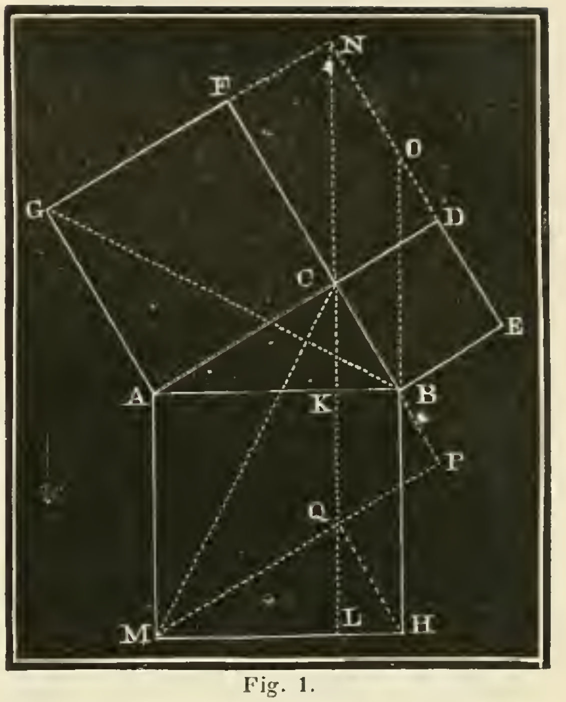


Fig. 1.

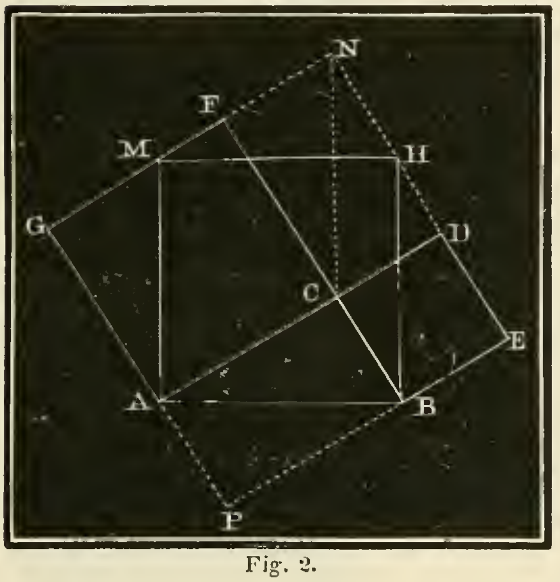


Fig. 2.

Dem. 2. Fig. 2. YOUNG.

$$ABHM = ABENG - (BEH + HMN + MGA)$$
$$= ABENG - (ABC + CDNF)$$
$$= ACFG + CBED.$$

But $\quad ABC + CDNF = BEH + HNM + MGA.$

$\therefore \quad ABHM = ACFG + CBED.$

Dem. 3. Fig. 1. YOUNG. $LHBK = BONC = BCDE.$

Dem. 4. Fig. 3. LEGENDRE. From the similar triangles ABC, ACD, BCD, we have $AC^2 = AB.AD$, $BC^2 = AB.BD$. $\therefore AC^2 + BC^2 = AB(AD + BD) = AB^2$.

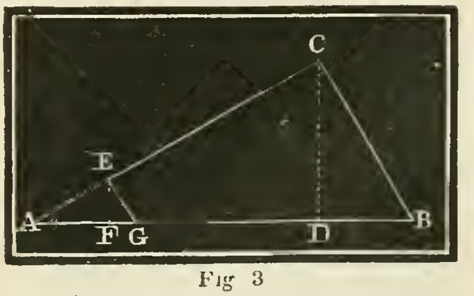


Fig 3

Dem. 5. Fig. 3. BEZOUT. OLIVER, Mathematical Monthly, Vol. I., p. 10. The triangles ABC, ACD, BCD, being similar, are to each other as the squares described upon their homologous sides, AB, AC, BC; but $ABC = ACD + BCD$. $\therefore AB^2 = AC^2 + BC^2$.

Dem. 6. Fig. 2. Knapsen.

$$ABHM = PENG - (APB + BEH + HNM + MGA)$$
$$= PENG - (ACBP + CDNF) = ACFG + BCDE,$$

since $APB + BEH + HNM + MGA = ACBP + CDNF$.

Dem. 7. Hoyt. See Mathematical Monthly, Vol. I., p. 231; also, p. 361.

Dem. 8. Pappus. Young. See also Mathematical Monthly, Vol. I., p. 354.

Dem. 9. Wheeler. See Mathematical Monthly, Vol. I., p. 159.

Dem. 10. Wheeler. See Mathematical Monthly, Vol. I., p. 160.

Dem. 11. Fig. 4. $MRCA = MLKA = ACFG$;
$BCRH = BKLH = BCDE$.
$\therefore\ ABHM = ACFG + BCDE$.

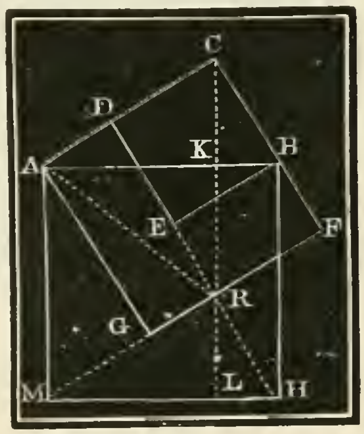


Fig. 4.

Dem. 12. Fig. 2. $ACNM + BCNH = ABHM$.
But $ACNM = ACFG$, and $BCNH = BCDE$.
$\therefore\ ABHM = ACFG + BCDE$.

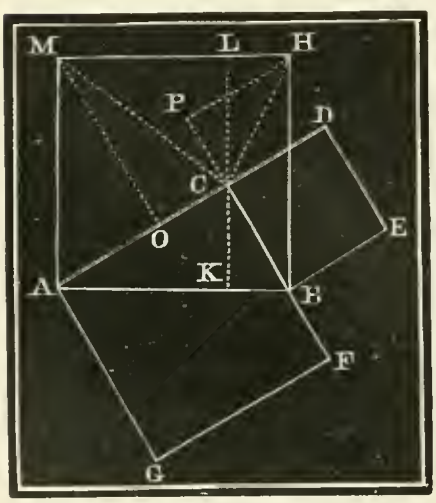


Fig 5.

Dem. 13. Fig. 5.
$ACM = \frac{1}{2} AKLM = \frac{1}{2} ACFG$,
and $BCH = \frac{1}{2} KBHL = \frac{1}{2} BCDE$.
$\therefore\ ABHM = ACFG + BCDE$.

Dem. 14. Fig. 6.
$ANC = \frac{1}{2} ACGH$
$= \frac{1}{2} AMLN$;

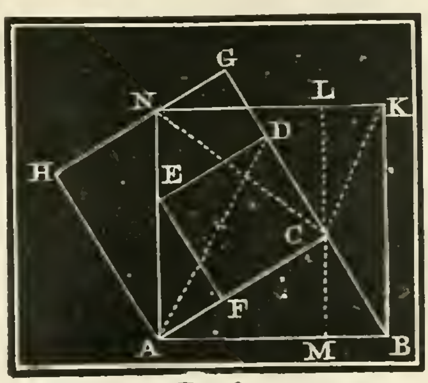


Fig. 6.

$AED = \frac{1}{2} CDEF = BCK = \frac{1}{2} MBLK$.
$\therefore\ ABKN = ACGH + CDEF$.

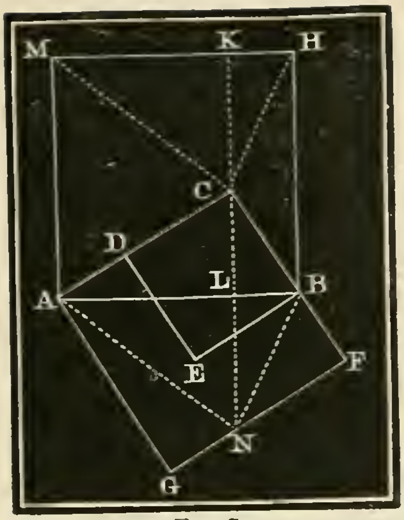


Fig 7.

Dem. 15. Fig. 7.

$ALKM = AMCN = 2\,ACN = ACFG$;

$BHCN = LBHK = 2\,BCN = BCDE$.

$\therefore ABHM = ACFG + BCDE$

Dem. 16.

Fig 8.

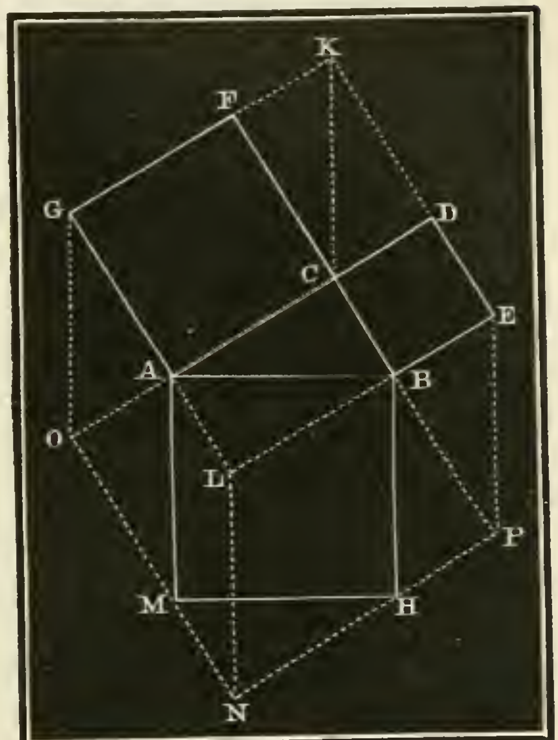


Fig. 8.

$OCKG + CPEK = ONPC$.

$CDKF = BEPH = AGOM$

$= 2\,ABC = 2\,HMN$.

$\therefore ABHM = ACFG + BCDE$

Dem. 17. Fig. 1.

$ACQM = AKLM = ACFG$;

$BCQH = BKLH = BCDE$.

$\therefore ABHM = ACFG + BCDE$.

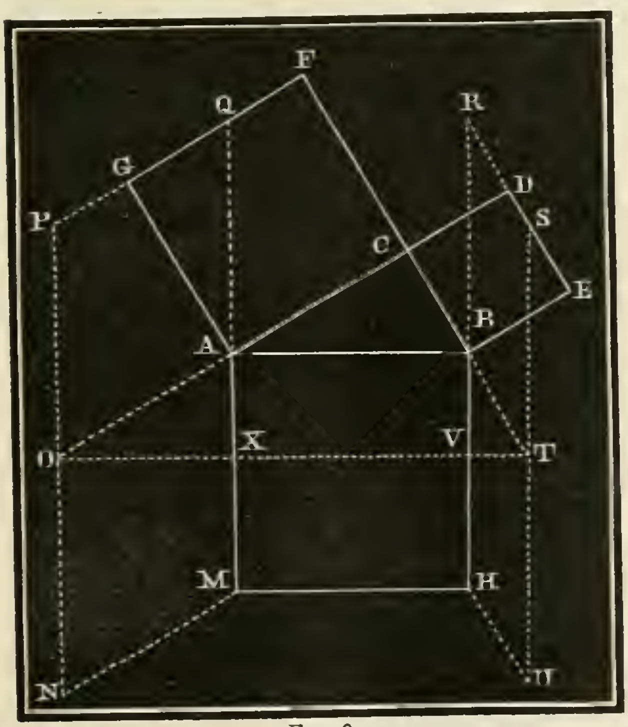


Fig 9.

Dem. 18. Fig. 8.

$ELGK = CONP$;

$CDKF = ACBL = 2\,ABC$.

$\therefore ABHM = ACFG + BCDE$.

Dem. 19. Fig. 9.

$AO = AC$, $BT = CB$;

$AOMN + BTUK = ABHM$; but

$AOMN = AOPQ = ACFG$,

$BTUH = BTSR = BCDE$.

$\therefore ABHM = ACFG + BCDE$.

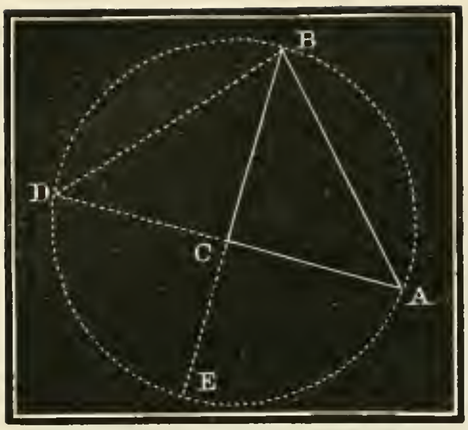


Fig 10.

Dem. 20. Fig. 10. YOUNG. This is an application of Prop. XXXII., Book IV. LEGENDRE.

$$CD = CA, \text{ and}$$

$$AB \times BD = BC^2 + AC \times CD.$$

$$\therefore \quad AB^2 = BC^2 + AC^2.$$

Dem. 21. Fig. 10. Application of Prop. XXXI., Book IV. LEGENDRE.

$$AB \times BD = BE \times BC = BC^2 + (BC \times CE = AC \times CD).$$

$$\therefore \quad AB^2 = BC^2 + AC^2$$

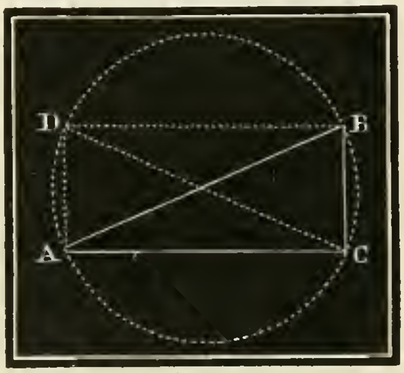


Fig. 11.

Dem. 22. Fig. 11. Application of Prop. XXXIII., Book IV. LEGENDRE.

$AB \times CD = AC \times BD + BC \times AD$, or

$$AB^2 = AC^2 + BC^2.$$

Dem. 23. Fig. 3. Take $AE = 1$,

$$CD = \frac{AD \times FE}{AF},\ BD = \frac{CD \times FE}{AF} = AD \times \frac{FE^2}{AF^2}.$$

$$\therefore \quad AB = AD + BD = AD + AD \times \frac{FE^2}{AF^2} = AD\left(1 + \frac{FE^2}{AF^2}\right);$$

or $AB = AD\left(\frac{AF^2 + FE^2}{AF^2}\right)$. But $AB = \frac{AC}{AF}$, and $AC = \frac{AD}{AF}$; hence, $AB = \frac{AD}{AF^2}$. Comparing these two values of AB, we get $AF^2 + FE^2 = 1$. Again, $AB \times AF = AC$, and $AB \times FE = BC$. Squaring these and adding

$$AB^2(AF^2 + FE^2) = AB^2 = AC^2 + BC^2.$$

Dem. 24. Fig. 3. Take $AG = 1$. If AB^2 is not equal to $AC^2 + BC^2$, let $x^2 = AC^2 + BC^2$; then

$$x = (AC^2 + BC^2)^{\frac{1}{2}} = AC\left(1 + \frac{BC^2}{AC^2}\right)^{\frac{1}{2}} = AC\left(1 + \frac{EG^2}{AE^2}\right)^{\frac{1}{2}}$$

$$= AC\left(\frac{AE^2 + EG^2}{AE^2}\right)^{\frac{1}{2}} = \frac{AC}{AE}$$

by Dem. 23. Therefore $AB^2 = x^2 = AC^2 + BC$.

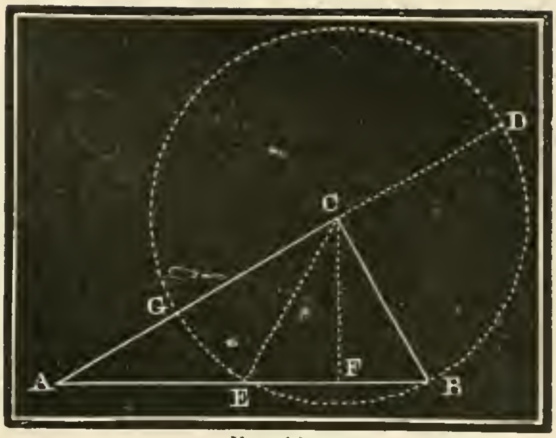


Fig 12

Dem. 25. Fig. 12. Application of Prop. XXIX., Book IV. LEGENDRE. $AE \times AB = AG \times AD$,

or $(AB - 2BF)AB$

$= (AC + BC)(AC - BC)$

$\therefore AB^2 = AC^2 - BC^2 + 2BF . AB$

$= AC^2 + BC^2$.

Dem. 26. Fig. 13. Application of Prop. XXX., Book IV. LEGENDRE.

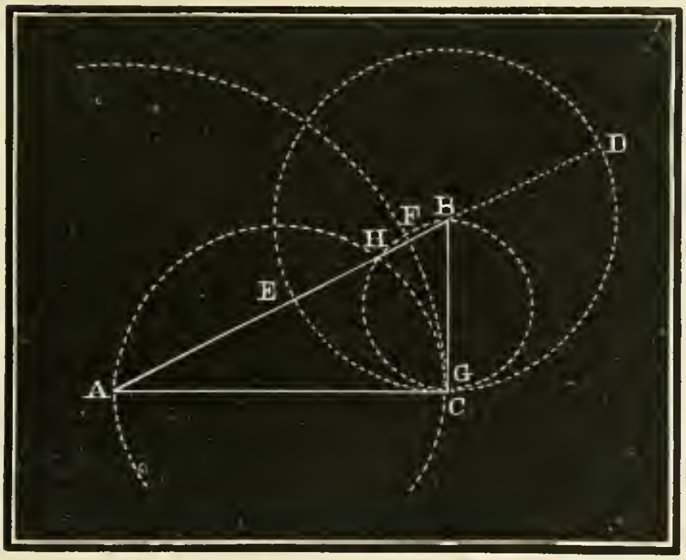


Fig 13.

$AC^2 = AE \times AD$

$= AE \times AB + AE \times BC$.

$AC^2 + BC^2 = AE \times AB$

$+ AE \times BC + BC^2$

$= AE \times AB + BC \times AB$

$= AB(AE + BC) = AB^2$; or

$AC^2 = AE \times AD$

$= (AB - BC)(AB + BC)$

$= AB^2 - BC^2$.

$\therefore AB^2 = AC^2 + BC^2$.

Dem. 27. Fig. 13. Application of the same proposition.

$$AC^2 = AH \times AB,\ BC^2 = BH \times AB.$$

Adding, we get $AC^2 + BC^2 = AH \times AB + BH \times AB = AB^2$.

Dem. 28. Fig. 13. Application of the same proposition.

$$AC^2 = AE(AB + BC), BC^2 = BF(AB + AC).$$

NOTE. Several of these demonstrations had already been received in nearly the same form; Dem. 11, from PLINY EARLE CHASE, Philadelphia; Dem. 22, from C. M. RUNK, Allentown, Pa.; Dem. page 361, Vol. I., JOSEPH FICKLIN, Jr., Trenton, Mo.; Dem. 26, second form, C. J. KEMPER, Harrisonburg, Va., and Prof. CHARLES A. YOUNG, Hudson, Ohio. The student will remember that the propositions upon which the Pythagorean is made to depend, must be proved independently of the latter.

Adding we get

$$
\begin{aligned}
AC^2+BC^2 &= AE\,(AB+BC)+BF\,(AB+AC)\\
&= AE\times AB+AE\times BC+BF\times AB+BF\times AC\\
&= AB(AE+BF)+AE\times BC+BF\times AC+BF\times AB-BF\times AB\\
&= AB(AE+BF)+AE\times BC+BF(BF+2AC)-BF\times AB\\
&= AB(AE+BF)+AE\times BE+BE^2-BF(AE+EB)\\
&= AB\,(AE+BF)+(AE+EB)\,(BE-BF)\\
&= AB\,(AE+BF)+AB\times EF=AB\,(AE+EF+FB)=AB^2.
\end{aligned}
$$

NOTE "ON THE HORIZONTAL THRUST OF EMBANKMENTS."

BY F. W. BARDWELL,
Professor of Mathematics in Antioch College, Yellow Springs, Ohio.

In an article "On the Horizontal Thrust of Embankments," in Vol. I., p. 175, I find what seems to me to be an error, which makes the principal formula defective, and therefore affects the conclusion. Without repeating the preliminary explanation, I will simply quote the portion which seems to be wanting in accuracy.

"The variable forces, F' and Q, may be resolved into the components $F'\cos v$, $Q\sin v$, perpendicular to af, and $F'\sin v$, $Q\cos v$, parallel to af. The force, $F\sin v$, acting along af upwards, must, assisted by the friction due to the normal pressure on af, just equal the parallel and opposite force $Q\cos v$; that is,

(1) $\quad F'\sin v+(F'\cos v+Q\sin v)f=Q\cos v.$"

But the normal pressure is $Q\sin v$ only, and equation (1) should be

$$F'\sin v+Qf\sin v=Q\cos v.$$

This corresponds to the formula usually given. I will only add that no allowance is made in the article referred to for the cohesion of particles.

NOTE ON DOUBLE POSITION.

BY REV. THOMAS HILL,
President of Antioch College, Yellow Springs, Ohio.

In the Cambridge Miscellany, Prof. Peirce expressed his regret that the old "Rule of False" had been omitted from the more recent elementary treatises on Arithmetic, published in this country. Since the date of that note, several Arithmetics have been published, containing the rule,—introduced however, as it were, timidly, and in a corner of the book. Yet the fundamental idea of the rule of double position lies at the basis of every attempt of the human intellect to discover truth. We frame an hypothesis, compare it with facts, and note the amount of discrepancy; then, for the corrections of our hypothesis, assume that the errors of our results are in some measure proportional to the errors of our data.

Now in mathematics, this gives a practical mode of attaining numerical results. Thus, let x be an unknown number, required to produce a fixed result, a. Let x' and x'' be two supposed values of x, which lead to the fixed results, $a+e$, and $a+e'$. Now, according to the general assumption of proportionality of errors in results to those in the data,

$$e : e' = x' - x : x'' - x.$$

And this leads to the value of x,

$$x = \frac{e\,x'' - e'\,x'}{e - e'}.$$

The identity of this formula with the following rule, given in Daboll's Arithmetic, is manifest.

"Multiply the first position by the last error, and the last position by the first error. If the errors are alike, [that is, both results too great, or both results too small,] divide the difference of the products by the difference of the errors, and the quotient will be

the answer. If the errors are unlike, divide the sum of the products by the sum of the errors, &c."

The advantage of this form of the rule is the ease with which it disposes of the signs of the errors. If the problem is an equation of the first degree, the answer is at once obtained with perfect accuracy; if not, the process must be repeated. For example: What is the side of a cube when the inches of superfices exceed the inches of solidity by 20?

Suppose a side of 2 inches. Then $6\,(2)^2 - 2^3 = 16$ and $20 - 16 = 4$. Suppose a side of 3 inches. Then $6\,(3)^2 - 3^3 = 27$ and $20 - 27 = -7$. Hence, by the rule, the approximate side is $(2 \times 7 + 3 \times 4) \div (7 + 4) = 2.36$.

Taking, therefore, the two positions, 2.3 and 2.4, we have $6\,(2.4)^2 - (2.4)^3 = 20.74$ and $20 - 20.74 = -.74$,

$$6\,(2.3)^2 - (2.3)^3 = 19.57 \text{ and } 20 - 19.57 = .43.$$

Then by the rule

$$[(.74 \times 2.3) + (.43 \times 2.4)] \div (.43 + .74) = 2.336 +.$$

Taking the new positions 2.336 and 2.337,

$$6\,(2.336)^2 - (2.336)^3 = 19.994\,; \text{ and } 20 - 18.994 = .006$$

$$6\,(2.337)^2 - (2.337)^3 = 20.0058\,; \text{ and } 20 - 20.0058 = -.0058.$$

Hence, for a nearer approximation, the side equals

$$[(.006 \times 2.337) + (.0058 \times 2.336)] \div (.006 + .0058) = 2.336508.$$

A simple rule, but requiring more care in the signs, consists in making the correction of the first position the unknown quantity. By the doctrine of proportions we readily obtain, from the given proportion, the following value of this correction,

$$x' - x = \frac{e\,(x' - x'')}{e - e'}.$$

That is to say, *Multiply the error of the first result by the difference of*

the positions, and divide by the difference of the results; the quotient is a correction to be added to the first position, if the first result shows it to be too small, and to be subtracted from the first position if the first result shows it to be too large.

This rule we prefer to the old rule quoted from Daboll, although it gives, of course, the same result.

ON THE INDETERMINATE ANALYSIS.

By Rev. A. D. Wheeler, Brunswick, Maine.

[Continued from Page 25.]

Proposition VI. If a and b be prime to each other, the indeterminate equation, $ax - by = c$, is always possible; and will admit of an infinite number of positive integral solutions.

Demonstration. Transferring and dividing, we have $x = \frac{c + by}{a}$, which has already been shown to be possible. Prop. V., Cor. 2.

Now as the remainders, resulting from the division of $c + by$ by a, recur in periods, and the number of periods is unlimited, since y may have all possible values, it follows that the number of solutions must also be unlimited; that is, infinite.

Prop. VII. If, in the equation $ax \pm by = c$, we call the least integral value of x, v; the next greater value of x, when the equation admits of more than one solution, will be $v + b$, the next $v + 2b$, and so on in Arithmetical progression.

Dem. The least value of x being v, and d denoting the difference, we shall have $v + d$ for the next greatest. Then $ax \pm by$ becomes $av \pm by' = c$ in the first case, or $y' = \frac{c - av}{\pm b} =$ an integer. In the second case, it becomes $av + ad \pm by'' = c$, or $y'' = \frac{c - av - ad}{\pm b}$ $=$ an integer. Therefore $y'' - y' = \frac{c - av - ad}{\pm b} - \frac{c - av}{\pm b} = \frac{- ad}{\pm b}$.

(Ax. 2). Therefore $\frac{d}{b}$ is an integer. (Prop. III.) But this can be an integer only when $d = b$, $d = 2b$, &c., which was to be proved.

Prop. VIII. The equation $ax + by = c$ is always possible for n solutions, when $c > nab$.

Dem. Let $c = nab + r$. Then we have

$$ax + by = nab + r; \text{ or}$$

$$ax - nab = -by + r; \text{ or}$$

$$x - nb = \frac{-by + r}{a} = -\frac{by - r}{a} = \text{an integer},$$

(Prop. V. Cor. 1). Therefore the equation admits of at least one solution.

Let $x = v$ for its first value; $x = v + b$ for its second; $x = v + 2b$ for its third; and so on, (Prop. VII). Then we shall have $x = v + (n-1)b$, for its nth value; and substituting this for x, we have

$$av + (n-1)ab + by = nab + r; \text{ or}$$

$$av + by = ab + r; \text{ or}$$

$$av - ab = -by + r; \text{ or}$$

$$v - b = -\frac{by - r}{a}, \text{ the same as before.}$$

Therefore the equation $ax + by = c$, will always admit of n solutions when $c > nab$.

Prop. IX. The equation $ax + by = c$ is impossible, in positive whole numbers, in the following cases. (1.) When a, or b, is prime to c, but not prime to each other. (2.) When $c < a + b$. (3.) When $c = ab$. (4.) When $c = ab - (ax' + by')$.

Dem. *Case* 1. Since a and b are supposed to have a common factor which is not in c, it is obvious that one member of the equation is divisible by it, while the other is not.

Case 2. The smallest integral value which can be given to x or y is 1. Giving to them this value, the equation becomes $a + b = c$. Consequently if c is less than $a + b$, the solution is impossible.

Case 3. Let $c = ab$; then we have

$$ax + by = ab$$

Now as a will divide one member of the equation, it must also divide the other, (Prop. I.); and consequently must divide by. But it cannot divide b, because it is prime to it. Therefore it must divide y, (Prop. III). Hence, $y = a$ is its least integral value. Substituting this value in the place of y, we have $ax + ab = ab$, or $ax = 0$. Wherefore $x = 0$; and the conditions of the equation cannot be fulfilled.

Case 4. Let $c = ab - (ax' + by')$.

Then we have $ax + by = ab - (ax' + by')$, or transposing,

$$a(x + x') + b(y + y') = ab.$$

Whence as before, $y + y$ will equal a, for its least value; and $x + x'$ will become zero. This case is therefore proved.

(To be continued.)

COMPLETE LIST OF DR. BOWDITCH'S WRITINGS.

To the various volumes of the Transactions of the American Academy of Arts and Sciences, Dr. Bowditch communicated the following memoirs:

Vol. II., Part II., Published in 1800.

1. *A New Method of Working a Lunar Observation.*

The object of this method was to establish a uniform rule for the application of corrections, so that there should be no variation of cases resulting from the distance and altitude of the observed bodies.

Dr. Bowditch says of this method, in a note, that "it was written several years ago, and before the publication of the Transactions of the Royal Society for 1797, in which is inserted a method, somewhat similar, invented by Mr. Mendoza y Rios. An appendix to the New Practical Navigator has lately been published, in which the corrections are all additive, and the work is shorter." It is particularly noticed and commended in the Connoissance des Tems (1808) then published under the direction of M. Delambre.*

* Zach (Corr. Astron. Vol. VI., p. 553, A. D. 1822), says: "M. Bowditch dans son *New American Practical Navigator* a aussi donné pour la réduction des distances lunaires une nouvelle méthode

Vol. III., Part I., Published in 1809.

2. *Observations on the Comet of* 1807. [pp. 1–18.]

3. *Observations on the Total Eclipse of the Sun, June* 16, 1806, *made at Salem.* [pp. 18–23.]

In a note to this communication, Dr. BOWDITCH makes, as is believed, the first public mention of an error in LAPLACE'S *Mécanique Céleste,* in the estimate of the oblateness of the earth, as calculated from the length of pendulums; showing that LAPLACE'S result ought to have been, upon his own principles, $\frac{1}{315}$ instead of $\frac{1}{336}$.

4. *Addition to the Memoir on the Solar Eclipse of June* 16, 1806. [pp. 23–33].

5. *Application of* NAPIER'S *Rules for Solving the Cases of Right-Angled Spheric Trigonometry to several Cases of Oblique-Angled Spheric Trigonometry.* [pp. 33–38].

This communication so alters NAPIER'S rules, as to make them include most of the cases of oblique-angled spheric trigonometry, and is marked by the same neatness, elegance, and simplicity, which characterized his first communication. These rules are now familiarly known in the text-books of Harvard College as "BOWDITCH'S Rules."

Vol. III., Part II., Published in 1815.

6. *An Estimate of the Height, Direction, Velocity, and Magnitude of the Meteor that exploded over Weston, in Connecticut, December* 14, 1807. [pp. 213–237].

This communication is of a very interesting character, and it rests upon numerous observations collected with great labor and assiduity. Dr. BOWDITCH considers the meteor in question to have had a course about eighteen miles above the earth, a velocity of more than three miles a second, and a probable cubic bulk of six millions of tons, — which others have estimated to be the contents of the pyramid of Cheops.*

7. *On the Eclipse of the Sun of September* 17, 1811, *with the Longitudes of several Places in this Country, deduced from all the Observations of the Eclipses of the Sun, and Transits of Mercury, and Venus, that have been published in the Transactions of the Royal Societies of Paris and London, and the Philosophical Society held at Philadelphia, and the American Academy of Arts and Sciences.*† [pp. 255–305.]

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abrégée, avec des tables, qui mérite d'être plus connue; aucun auteur Européen n'en a encore parlé; il vient de la perfectionner dans sa quatrième édition stéréotpye publiée à New York en août 1817. Nous la recommandons à l'attention des professeurs et auteurs des traités de navigation." In Vol. X., p. 321, A. D. 1824, he says: "La méthode de M. BOWDITCH a l'avantage sur toutes les autres méthodes d' approximation, que toutes les corrections sont toujours additives, et qu'on n'a jamais besoin de faire attention à des cas particuliers; les règles sont générales;" and proceeds to give a detailed account of it. — See also note to article 15.

* The *Zeitschrift für Astronomie,* Vol. I., p. 37, A. D. 1816, gives the results arrived at in this communication, and calls it "einer interressanten Arbiet."

† The *Zeitschrift für Astronomie,* Vol. I., p. 90, 1816, mentions the observations of the eclipses of the sun, June 16, 1806, and September 17, 1811, as contained in these volumes, &c., and states that "BOWDITCH hat den grössern Theil davon zu Längenbestimmungen benutzt und zugleich dabey, für eine Menge Amerikanischer Orte, Hülfsgrössen zur leichtern Berechnung des Nonagesimus gegeben;"

8. *Elements of the Orbit of the Comet of* 1811. [pp. 313—326].

In this, as in his second communication, he arrives at his results after almost incredible labor, rendered necessary by the want of the improved methods of the present day.* The original volume, containing his calculations in the case of this latter comet, now preserved in his library, contains one hundred and forty-four pages of close figures, probably exceeding one million in number, though the result of this vast labor forms but a communication of twelve pages.†

9. *An Estimate of the Height of the White Hills in New Hampshire.* [pp. 326—328].

10. *On the Variation of the Magnetic Needle.* [pp. 337—344].

This communication, in like manner, which is of quite an interesting character, and of considerable practical importance, was the result of five thousand and twenty-five observations, during a period of five years.

11. *On the Motions of a Pendulum suspended from two points.* [pp. 413—437].

This communication is also one of interest and value; and the little wooden stand, from which a leaden ball was suspended, still exists, to remind us of the zeal and assiduity with which Dr. Bowditch watched the various curves and lines which the ball described.‡

12. *A Demonstration of the Rule for finding the Place of a Meteor, in the Second Problem, page* 218 *of this Volume.* [pp. 437—439]

Vol. IV., Part I., Published in 1818.

13. *On a Mistake which exists in the Solar Tables of Mayer, Lalande, and Zach.*§

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and Zach, in his Corr. Astron. Vol. X., p. 494, A. D. 1824, has a table of the longitudes and latitudes of places determined by astronomical observations calculated by Dr. Bowditch.

* See Dr. Bowditch's letter (Zach, Corr. Astron. Vol. X., p. 228), before referred to, where this fact is stated. The editor, in page 248, gives the elements of the orbits of the comets calculated by Dr. Bowditch wholly from American Observations.

† Mr. Encke, in speaking to a friend of Dr. Bowditch, at Berlin, in 1836, said that he had known him from the time when this paper appeared; and that he had never seen an American since, without asking him what he could tell him about its author; and the *Zeitschrift für Astronomie*, Vol. I., p. 44, gives an account of this communication "von dem Amerikanischen Astronomen Bowditch."

‡ This subject is mentioned is his letter to Baron Zach, before alluded to, (*Corr. Astron.* Vol. X., p. 227). The editor, in his note, p. 246, says the remarkable variety of the motions of a pendulum thus suspended, and the very curious experiments of Prof. Dean, who explains, in this mode, the apparent motion of the earth as seen from the moon, engaged Dr. Bowditch in the examination of the theory of these motions. The result has been, he adds, "une recherche très intéressante." "Comme ce mémoire mérite d'être mieux connu, et qu'il ne l'est pas généralement, vu la difficulté de se procurer des livres Américains, nous en donnerons la traduction dans un de nos cahiers."

§ Dr. Bowditch states, that "The attraction of Jupiter produces an equation in the expression of the Sun's distance from the earth, and a Table is given for its computation, by Mayer, in 1770," &c., "and ever since this table was first published, which is about fifty years, an error of six signs has always existed in the argument, by which the correction is found; so that, when the equation is really *subtractive*, it will frequently be found by the table to be *additive*, and the contrary." "In De Lam-

14. *On the Calculation of the Oblateness of the Earth, by means of the observed Lengths of a Pendulum in different Latitudes, according to the Method given by Laplace, in the Second Volume of his "Mécanique Céleste;" with remarks on other parts of the same Work relating to the Figure of the Earth.* [pp. 3—24.]

The object of this communication is to correct certain errors in the article "Earth," in Ree's Cyclopædia, to the end that currency should not be given to inaccurate ideas on the subject, by that popular work.

15. *Method of Correcting the apparent Distance of the Moon from the Sun, or a Star, for the Effects of Parallax, and Refraction.* [pp. 24–31.]

This is but the rule given in the Practical Navigator, making all the corrections in question additive. It is another instance of the simplicity at which he always aimed in his rules and formulas.*

16. *On the Method of computing the dip of the Magnetic Needle in different Latitudes, according to the Theory of Mr. Biot.* [pp. 31—36.]

17. *Remarks on the Methods of correcting the Elements of the Orbit of a Comet, in Newton's "Principia" and in Laplace's "Mécanique Céleste.* [pp. 36—48.]

This communication proves that two equations in the Principia, the accuracy of which several commentators upon that work had attempted to prove, and as to which no doubts had yet been expressed or insinuated, always made the corrections in question "double of what they ought to be," and restricts the method of Laplace as appropriate only where the number of observations is small.

18. *Remarks on the usual Demonstration of the Permanency of the Solar System, with respect to the Eccentricities and Inclinations of the Orbits of the Planets.* [pp. 48—51.]

19. *Remarks on Dr. Stewart's Formula for computing the Motion of the Moon's Apsides, as given in the Supplement to the Encyclopædia Britannica.* [pp. 51—61.]

This is a very curious and interesting communication. A method which, notwithstanding doubts had been expressed respecting it, had been sanctioned as accurate by

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bre's Solar Tables, published in 1806, the form of the table is wholly altered, the method of entry by a double argument being used; and by thus taking a different path, the error is avoided, without noticing that it really does exist in the other works."

Baron Zach, in his *Monatliche Correspondenz,* Vol. VIII., p. 449, A. D. 1803, says that Bowditch, an American astronomer, has called attention to this mistake; and, after admitting its importance, frankly adds, "Allen Astronomen, welche sich mit Verfertigung der Sonnen-Taflen beschäftigt haben, einen La Caille, Tob. Mayer, La Lande, De Lambre und *mir* ist dieser Fehler entgangen."

* In Zach's *Monatl. Corres.,* Vol. XVII., p. 411, A. D., 1808, this method is mentioned as being in the Appendix to the New American Practical Navigator, printed in Newburyport, 1804; and the editor says: "Der Verfasser ist ein Americaner, Bowditch, und De Lambre hat es der Mühe werth gehalten eine umständliche Darstellung dieses Verfahrens zu geben." Then follows a somewhat minute account of the method. See note to Article I.

Dr. Hutton, by Lalande, and Playfair,—the latter of whom even considered its accuracy to have been demonstrated,—is in this memoir proved to have been true only in the particular case supposed; and it is shown that, as a general method, it wholly fails.

Vol. IV., Part II., Published in 1820.

20. *On the Meteor which passed over Wilmington, in the State of Delaware, November* 21, 1819. [pp. 3—14.]

21. *Occultation of Spica by the Moon, observed at Salem.* [p. 14.]

22. *On a Mistake which occurs in the Calculation of Mr. Poisson relative to the Distribution of the Electrical Matter upon the Surfaces of two Globes, in Vol. XII., of the Mémoires de la classe des sciences mathématiques et physiques de l' Institut Impérial de France.* [pp. 15—17.]

23. **Elements of the Comet of* 1819. [pp. 17—19.]

Besides the above contributions to the Memoirs of the American Academy, Dr. Bowditch was the writer of several other articles, among which may be mentioned the following:

1. *Notice of the Comet of* 1807. Published in the Monthly Anthology, for December, 1807, Vol. IV. [pp. 653—654.]

2. *Review of a "Report of the Committee (of Congress) to whom was referred, on the 25th January,* 1810, *the Memorial of William Lambert, accompanied with sundry Papers relating to the establishment of a First Meridian for the United States, at the permanent seat of their Government.* Published in the Monthly Anthology, for October, 1810, Vol. IX. [pp. 245—266.]

This article occupies twenty-one pages, and proves very conclusively the great advantages of continuing to estimate the longitude from Greenwich, which Mr. Lambert considered "a sort of degrading and unnecessary dependence on a foreign nation," and "an incumbrance unworthy of the freedom and sovereignty of the American people." This Memorial the reviewer shows to be a compilation, with needless repetitions and palpable mistakes, evincing a great want of knowledge in the principles of the calculations;" and that, "both as respects its object and execution, it was wholly undeserving the patronage of the National Legislature."

3. *Defence of the Review of Mr. Lambert's Memorial.* Published in the Monthly Anthology, for January, 1811, Vol. X. [pp. 40—49.]

Mr. Lambert having made an angry reply, charging his reviewer with "twistical cunning," "ingenious quibbling," "zeal for the honor of the British nation, and convenience of British mariners," and challenging him "to examine his computation of the longitude of the Capitol at Washington from Greenwich, *and to point out a mistake that*

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* For a statement of Dr. Bowditch's communications to the Memoirs of the Academy, and an abstract of their contents, from which several of our remarks in the text are condensed, see Mr. Pickering's Eulogy, [pp. 17-31].

can be made palpable." — Dr. Bowditch, in his reply, considers these charges of Mr. Lambert as beneath his notice, but accepts his challenge, and proves *that there is an error in every one of the six examples he has given.*

These two papers were fatal to the proposed project; and fortunately for the interests of science, Greenwich continues to be the first meridian of all who speak the English language.

4. *Review of a Treatise on the most easy and convenient Method of computing the Path of a Comet, from several Observations; by William Olbers, M. D., Weimar,* 1797;" *and of* "*Theoria Motus Corporum Cœlestium in Sectionibus Conicis Solem ambientium;*" by Charles Frederick Gauss; Hamburg, 1809. Published in the North American Review, for April, 1820, Vol. X. [pp. 260—272.]*

This article gives an account of several German astronomers and their most noted periodical publications. Thus it contains a notice of Dr. Olbers — "the Columbus of the planetary world," — and of Gauss, the authors of the two works reviewed; an account of *Bode's Astronomisches Jahrbuch, Zach's Monatliche Correspondenz* and the *Zeitschrift für Astronomie.* It states the fact that, "out of *thirteen* primary planets and satellites, discovered since the year 1781, we are indebted to persons born in Germany for *twelve;* and that, in the determination of the orbits of these new bodies, they have done more than all the other astronomers in the world."

5. *Review of* "*A remarkable astronomical Discovery, and Observations of the Comet of July,* 1819; *by Dr. Olber's, of Bremen; published in Bode's Astronomisches Jahrbuch for* 1822; [and of two other articles in the same work, for the years 1822 and 1823, on the same subject, by Professor Encke of the Ducal Observatory at Seeberg, near Gotha.] Published in the North American Review, for January, 1822, Vol. XIV. [pp. 26—34.]

A copy of this article on Encke's Comet Dr. Bowditch also sent with the letter before mentioned to Baron Zach, who, in his notes, states that Dr. Bowditch, "has here collected all that has been said and done respecting this famous comet." In the concluding paragraph of this article, the reviewer expresses his regret that, "while Great Britain alone can boast of more than thirty public and private observatories

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* A copy of this article Dr. Bowditch sent to Baron Zach, with the letter before referred to, *marking a part of it as written by Mr. Everett, the editor.* Zach publishes extracts from it in his notes upon this letter, Vol. X., p. 231, and says: "It will be interesting to the reader to learn how men of science in America render justice to those of Germany, while they reproach their bretheren beyond the water for the little attention which they have bestowed upon our productions." Dr. Bowditch mentions in this review an interesting paper which Mr. Ivory had published in the Transactions of the Royal Society of London, 1814, giving a method of his own for computing the orbit of a comet, which "upon examination turns out to be nothing more than that which Mr. Olbers had published in his work above seventeen years before, although this coincidence must have been wholly unknown to Mr. Ivory, and to the other members of the Royal Society. We consider this as a striking instance of the little attention paid in Great Britain to works of mathematical science printed in Germany." The passage added by Mr. Everett was merely that which states a like neglect of German *literature.*

of considerable note, we have not, in the whole United States, one that deserves the name." He also speaks of the duties imposed on the importation of mathematical instruments and scientific works, as *fines* and *penalties*, which had been justly called "a bounty on ignorance," &c. This whole paragraph is extracted by Zach, (Vol. X., p. 245), and he says, "Voici de quelle maniere un bon républicain exhales on chagrin en public; c'est au moins quelque chose, &c."

6. *Letter to Baron Zach, dated November* 22, 1822; *with a Postscript, dated December* 20, 1823. Published in his Correspondance Astronomique for the year 1824.* Vol. X. [pp. 223—230.]

7. *Review, entitled* "*Remarks on several Papers published in former volumes of this Journal;*" [the first being remarks on "A new Algebraic Series, by Professor Wallace, of Columbia, S. C.] published in Silliman's Journal for 1824, Vol. VIII. [pp. 131—139] — and *Remarks on Mr. Wallace's Reply,* published in the same Journal for 1825, Vol. IX. [pp. 293—304.]

The reviewer expresses his surprise that any offence should have been given by the mere statement of the *historical fact* that this "new series" was but the usual development of the binomial theorem, and the same which had been given by Euler fifty years before.†

8. *Review of* "*Fundamenta Astronomiæ,*" by Frederick William Bessel; 1818; of the tables of the Moon, by M. Buckhardt; 1812; — of the New Tables of

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* This letter has been already more than once referred to, and contains many interesting facts. The editor's comments upon it occupy twenty pages. With this letter Dr. Bowditch had sent, besides his articles mentioned in the two last preceding items, a copy of the fifth edition of the Practical Navigator. The editor says of him: "C'est le premier, et jusqu' à-present le seul grand géomètre en Amérique."

† Professor Wallace in his reply, states that he did not claim the series as *new,* and appeals to a reference which he had made in his original article to Mr. Stainville, &c., and not knowing who his opponent was, says that he does not, like his reviewer, refer his readers to the Complement des Eléméns d'Algèbre, *however useful as a school book,* &c. He also states, "that the results which Euler has given do not include a single case of a transcendent function, and were only given as examples of the applications of the simplest case of the binominal theorem," &c. Dr. Bowditch in his rejoinder mentions the vague terms in which Mr. Stainville had been originally referred to, and says, "It now appears that Mr. Stainville gave it as new for the *first* time in 1818, and Professor Wallace for the *second* time in 1824, Euler's having been published in 1775;" and again: "It is believed that most persons, after reading what Professor Wallace has written, would suppose he claimed some, if not a very large portion, for his own. But the real fact is, that *none* of it is *his.* The whole of the first seven pages, and a large portion of the two remaining pages, of Professor Wallace's first communication, are merely literal translations from Stainville and Gergonne; and what is not copied from them is quite unimportant." He also says: "It is a fact, *notwithstanding the positive* declaration of Professor Wallace to the contrary, that Euler's demonstration is not restricted to this very simple case, but is general for all values of the exponent, whether integer, fractional, negative, or surd; and it is characterized by Lacroix as being elegant and rigorous." This review will be found quite amusing and piquant. It is, like the articles on Mr. Lambert's Memorial, both as to matter and style, a fair specimen of Dr. Bowditch's powers as a controversial writer.

Jupiter and of Saturn, by M. BOUVARD; 1808; — of the Tables of the Satellites of Jupiter, &c., by M. DELAMBRE; 1817; — of the Tables of Venus, of Mars, and of Mercury, by B. DE LINDENAU; 1810, 1811 and 1813; — and of the Memoir on the Figure of the Earth, by M. DE LAPLACE; 1817 and 1818. *Published in the North American Review for April 1825, Vol. XX. [pp. 309—367.]†

This brief but most comprehensive article upon modern astronomy will be found to possess an uncommon degree of interest. It consists of a series of biographical sketches, in which are described all who have been remarkable for the successful cultivation of physical science in modern times, bringing into view their actual and relative services and merits, and awarding to each the degree of approval to which he was entitled; — the writer now dwelling with enthusiasm upon his favorite LAGRANGE, now bestowing a more qualified and guarded approbation, or a positive censure upon others inferior in powers and attainments to that distinguished mathematician or opposite to him in character.‡ It comprises especially a very full account of Dr. BRADLEY's observations, and of BESSEL's services in reducing them; of the best makers of mathematical instruments—GRAHAM, BIRD, RAMSDEN, TROUGHTON, JONES, REICHENBACH, FRAUENHOFER, HERSCHEL, &c., of the successive astronomers royal at Greenwich, and of the other chief European observers; and lastly, "it gives an account of the labors of those mathematicians who have improved the science of astronomy by their calculations of the effects of the mutual attractions of the heavenly bodies."

Mécanique Céleste, by the Marquis DE LAPLACE, Peer of France, Grand Cross of the Legion of Honor; Member of the French Academy; of the Academy of Sciences of Paris, of the Board of Longitude of France, of the Royal Societies of London and Göttingen, of the Academies of Sciences of Russia, Denmark, Sweden, Prussia, Holland, and Italy; Member of the American Academy ot Arts and Sciences, &c. Translated with a Commentary, by NATHANIEL BOWDITCH, LL.D., Fellow of the Royal Societies of London, Edinburgh, and Dublin; of the Astronomical Society of London, of the Philosophical Society held at Philadelphia, of the

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* The titles of the particular works reviewed are here given in an abridged form.

† In the notes to Mr. PICKERING's Eulogy, p. 95, a list is given, without comment, of six of the above eight articles, the fifth and seventh not being noticed. All these occasional publications of Dr. BOWDITCH, except the letter to Baron ZACH, were collected by him in two volumes, now in his library.

‡ Thus, he says, "Upon the decease of EULER, LAGRANGE remained indisputably the greatest mathematician then living," &c.; while of Dr. BRADLEY's successor he says: "Dr. BLISS was wholly unworthy of the office of Astronomer Royal. The account of his life by LALANDE, is comprised in less than a dozen words — BLISS était Astronome Royal: il mourut en 1765." This article is the one, of all Dr. BOWDITCH's occasional publications, which exhibits in the clearest light, his peculiar talents and acquirements. Evidently the work of one possessing a knowledge of the actual state of mathematical science, in its various departments, as extensive and minute as was possessed by any individual then living, — it is throughout a record of the most sound and impartial criticism. Any biography of him which has not this review in an appendix, must be incomplete.

American Academy of Arts and Sciences; Corresponding Member of the Royal Societies of Berlin, Palermo, etc. Vol. I., pp. 746, 1829. Vol. II., pp. 990, 1832. Vol. III., pp. 1017, 1834. Vol. IV., pp. 1018, 1839. Boston: From the press of Isaac R. Butts: Charles C. Little and James Brown, Publishers.

Dr. Bowditch was also, for many years, a contributor to the "Annalist" and "Mathematical Diary," solving every question there proposed, in his usual style of simple elegance. He also wrote or corrected various articles in the American edition of Rees's Cyclopædia. And all these various publications were the employment merely of those leisure hours which were left to him after all the calls of active business, and all the claims of social and domestic life, had been most fully answered and more than this, notwithstanding all the occasional scientific labors which have been mentioned, such was his wonderful economy of time, that, within the same period, he also completed what has justly been characterized as the gigantic undertaking of making the Translation and Commentary of *Mécanique Céleste*, a work upon which, almost exclusively, will rest his fame as a man of science.* Upon recurring to the Translator's Preface in the first volume, it will be found there stated that the notes were written at the time of reading the volumes, as they were successively published. The translation was made between the years 1814 [misprinted 1815] and 1817, at which time the four first volumes, with the several appendices and notes, were ready for publication." The fifth volume, published by Laplace, twenty years after the others, was never translated by Dr. Bowditch, though he wrote many important notes upon it.† It was his intention, however, had he lived, to translate the volume. Death has forever defeated that intention. The work which he had so nearly completed, no one lives to finish as he would have finished it; but like the beautiful painting, from which was taken the accompanying engraving, and which never received the final touch of the dying artist, it is more interesting from the circumstance under which it was left incomplete.

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* Baron Zach, in his *Correspondance Astronomique*, Vol. X., p. 234, A. D. 1824, says, "Nous finérons cette note par apprendre à nos lecteurs ce que nous a révélé le professeur Everett, que M. Bowditch a traduit en anglais toute la Mécanique Céleste de M. Laplace, avec un ample commentaire, mais qu'on n'a pu encore le persuader de publier cet ouvrage, qui ne pourrait que lui faire un honneur infini, ainsi qu' à son pays, mais nous soupçonnons qu' il attend pour cela l'ouvrage de MM. Plana et Carlini, qui est sur le métier, et qui ne tardera pas à paraitre." A similar public announcement of this fact had been made in the North American Review for April, 1820, Vol. X., p. 272; and the editor says that Dr. Bowditch "has not, however, yet been prevailed upon to do honor to himself and to his country, by the publication of so great and arduous a work."

† A day or two only before his death, he received from Europe a translation, executed by a young lady whom he had never seen, but who was soon to become his daughter, embracing in seventy manuscript pages, the first part of the fifth volume; — a suitable offering of filial duty to one who never lived to thank her in person for her kindness, but who left for her at his decease, an affectionate letter written exactly a week before his death.

This valuable *résumé* of the scientific and literary labors of Dr. Bowditch is taken from his Memoir, in the fourth volume of Mécanique Céleste, written by N. I. Bowditch, Esq.; to whose kindness the readers of the Monthly are indebted for the beautiful portrait of his distinguished father which graces this number. — Ed.

Mathematical Monthly Notices.

Lessons Introductory to the Modern Higher Algebra. By Rev. GEORGE SALMON, A. M., Fellow and Tutor, Trinity College, Dublin. pp. 147, 8vo. Dublin: HODGES, SMITH & Co., Grafton Street, Booksellers to the University. 1859.

IT is well known to our readers that during the last few years, there has grown up, mostly by the labors of English mathematicians, what may be called a new department of analysis, and what our author terms the "Modern Higher Algebra," or the "Algebra of Linear Transformations, a department of Analysis which has been created during the last twenty years." The papers upon different points in this new analysis, which come to us through "The Quarterly Journal of Pure and Applied Mathematics," "The Memoirs of the Royal Society," and the other various European Journals and Transactions, presuppose so much knowledge, that all, but the very few, have passed them by without making any effort to read them. It is therefore with the deepest satisfaction, that we are able to announce a purely elementary work, prepared by one whose ability to do justice to the undertaking is so fully shown in his "Conic Sections" and "Higher Plane Curves," as a "guide to this branch of Algebra." We must for the present content ourselves with promising, in future numbers of the MONTHLY, so much of the volume before us as will define the technical terms used in the new analysis, a list of which we append.

Bezoutiant.	Canonical Form.	Canonizant.
Congredient.	Contragredient.	Combinants.
Commutants.	Concomitant.	Mixed Concomitant.
Contravariant.	Covariant.	Determinant.
Dialytic.	Discriminant.	Eliminant.
Emanant.	Evectant.	Hessian.
Hyperdeterminant.	Invariant.	Jacobian.
Minor.	Quantic.	Reciprocal of a Determinant.
Reciprocal.	Substitutions.	Singular Roots.
Symmetrical Determinant.	Umbral.	Weight.

The term *quantic* denotes a homogeneous function in general, and the words *quadric*, *cubic*, *quartic*, *quintic*, *nic*, denote quantities of the 2nd, 3rd, 4th, 5th, nth degrees. Quantics are also distinguished into binary, ternary, quartenary, n-ary, according as they contain 2, 3, 4, n variables. There are other subsidiary terms which it is not necessary to give in this connection.

Our author says, "To A. CAYLEY, Esq., and J. J. SYLVESTER, Esq., I beg to inscribe this attempt to render some of their discoveries better known, in acknowledgment of the obligations I am under, not only to their published writings, but also to their instructive correspondence." We trust that this volume may be the means of introducing American students into this new and fertile field of research. We commend it to their attention.

A Treatise on Problems of Maxima and Minima, solved by Algebra. By RAMCHUNDRA, late Teacher of Science, Delhi College. Calcutta: Printed by P. S. D'ROZARIO & Co., Tank Square, 1850, pp. 180. Reprinted by order of the Honorable Court of Directors of the East India Company for circulation in Europe and in India, in acknowledgment of the merits of the author, and in testimony of the sense entertained of the importance of independent

speculation as an instrument of national progress in India. Under the superintendence of AUGUSTUS DE MORGAN, F. R. A. S., F. C. P. S., of Trinity College, Cambridge; Professor of Mathematics in University College, London. London: WM. H. ALLEN & Co., 7 Leadenhall St., 1859.

An extended notice of this interesting work will hardly be necessary, in addition to the examples already given of our author's method, on page 41. Chapter I. contains solutions of 55 problems, involving equations of the first and second degrees; Chapter II. 27 of the third degree; Chapter III. 18 of the fourth, fifth, sixth and seventh degrees; Chapter IV. 18, involving two or more variable quantities; the whole forming a most interesting collection of problems in maxima and minima, every one of which is solved by the methods used in the examples we have quoted. Prof. DE MORGAN's Preface is exceedingly interesting, and we regret not to be able to lay it in full before our readers. After giving some account of the correspondence which led the Court of Directors of the Honorable East India Company to concur in his views, the Editor says: "I shall at once proceed to a short account of these views; after which I shall give some account of RAMCHUNDRA, the author of the work. Of course it will be remembered that the late Court of Directors is in no way answerable for the details of my exposition, though their decided approbation was bestowed on the general sketch which I laid before them.

There are many persons, even among those who seriously turn their thoughts to the improvement of India, who look upon the native races as men to be dealt with in the same manner as Caffres or New Zealanders. Judging by the lower races of the Peninsula, and judging even these more by the grosser parts of their mythology than by the state of domestic life and hereditary institutions, they presume that the Indian question resolves itself into an inquiry how to create a mind in the country, and that mind fashioned on the English standard. They forget that at this very moment there still exists among the higher castes of the country — castes which exercise vast influence over the rest — a body of literature and science which might well be the nucleus of a new civilization, though every trace of Christian and Mahomedan civilization were blotted out of existence. They forget that there exists in India, under circumstances which prove a very high antiquity, a philosophical language which is one of the wonders of the world, and which is a near collateral of the Greek, if not its parent form. From those who wrote in this language we derive our system of arithmetic, and the algebra which is the most powerful instrument of modern analysis. In this language we find a system of logic and of metaphysics; an astronomy worthy of comparison with that of Greece in its best days; above comparison, if some books of PTOLEMY's Syntaxis be removed. We find also a geometry, of a kind which proves that the Hindoo was below the Greek as a geometer, but not in that degree in which he was above the Greek as an arithmetician. Of the literature, poetry, drama, &c., which flourished in union with this science, I have not here to speak.

Those who consult COLEBROOKE's translation of the *Vija Ganita*, or the account given of it in the *Penny Cyclopædia*, will see that I have not exaggerated the point most connected with this preface.

"Greece and India stand out, in ancient times, as the countries of indigenous speculation. But the intellectual fate of the two nations was very different. Among the Greeks, the power of speculation remained active during their whole existence as a nation, even down to the taking of Constantinople: it declined, indeed, but it was never extinguished. Their latest knowledge was inquisitive, as well as their earliest. They preserved their great writers unabridged and unaltered and EUCLID did not degenerate into what are called *practical rules.*

"In India, speculation died a natural death. A taste for *routine* — a thing to which inaccurate thinkers give the name of *practical* — converted their system into a collection of rules and results. Of this character are all the mathematical books which have been translated into English; per-

haps all which still exist. That they must have had an extensive body of demonstrated truths is obvious; that they lost the power and the wish to demonstrate is certain. The Hindoo became, to speak of the highest and best class, the teacher of results which he could not explain, the retailer of propositions on which he could not found thought. He had the remains of ancestors who had investigated for him, and he lived on such comprehension of his ancestors as his own small grasp of mind would allow him to obtain. He fed himself and his pupils upon the chaff of obsolete civilization, out of which Europeans had thrashed the grain for their own use.

"But the mind thus degenerated is still a mind; and the means of restoring it to activity differ greatly from those by which a barbarous race is to be gifted with its first steps of progress. No man alive can, on sufficient data, reason out the restoration of a decayed national intellect, possessed of a system of letters and science which has left nothing but dry results, inveterate habits of routine, great reverence for old teachers, and small power of comprehending the very teaching which is held in traditional respect. And this because the question is now tried for the first time. Many friends of education have proposed that Hindoos should be fully instructed in English ideas and methods, and made the media through which the mass of their countrymen might receive the results in their own languages. Some trial has been given to this plan, but the results have not been very encouraging in any of the higher branches of knowledge. My conviction is, that the Hindoo mind must work out its own problem; and that all we can do is to *set it to work;* that is, to promote independent speculation on all subjects by previous encouragement and subsequent reward. This is the true plan; all others are neither fish nor flesh.

"The history of England, as well as of other countries, having impressed me with a strong conviction that pure speculation is a powerful instrument in the progress of a nation, and my own birth and descent having always given me a lively interest in all that relates to India, I took up the work of Ramchundra with a mingled feeling of satisfaction and curiosity: a few minutes of perusal added much to both. I found in this dawn of the revival of Hindoo speculation two points of character belonging peculiarly to the Greek mind, as distinguished from the Hindoo; one of which may have been fostered by the author's European teachers, but certainly not the other.

"The first point is a leaning towards geometry. Persons who are not mathematicians imagine that all mathematicians are for all mathematics. Nothing can be more erroneous. Not merely have the two great branches, geometry and algebra, their schools of disciples, each of which looks coldly upon the other; but even geometry itself, and algebra itself, have subdivisions of which the same thing may be said. For example, Mr. Drinkwater-Bethune, above mentioned, was by taste an *algebraist;* as a practised eye would at once detect from his unfinished work on equations. Business brought him to my house one morning, nearly thirty years ago, at a time when I happened to be studying some of the geometrical developments of the school of Monge. On my pointing out to him some of the most remarkable of the conclusions, he said, with a smile, "I see that sort of thing has charms for you." Now the Hindoo was also an algebraist, as decidedly as the Greek was a geometer; the first sought refuge from geometry in algebra, the second sought refuge from arithmetic in geometry. The greatness of Hindoo invention is in algebra, the greatness of Greek invention is in geometry. But Ramchundra has a much stronger leaning towards geometry than could have been expected by a person acquainted with the *Vija Ganita;* but he has not the power in geometry which he has in algebra. I have left one or two failures — one very remarkable — unnoticed, for the reader to find out. Should this preface — as I hope it will — fall into the hands of some young Hindoos who are systematic students of mathematics, I beg of them to consider well my assertion that their weak point must be strengthened by the cultivation of pure geometry. Euclid must be to them what Bhascara, or some other algebraist, has been to Europe.

"The second point is yet more remarkable. Greek geometry, as all who have read EUCLID may guess, gained its strength by *striving against self-imposed difficulties.* It was not permitted to take instruments from every conception which the human mind could form; definite limitation of means was imposed as a condition of thought, and it was sternly required that every feat of progress should be achieved by those means, and no more. Just as the Greek architecture studied the production of rich and varied effect out of the simplest elements of form, so the Greek geometry aimed at the demonstration of all the relations of figure on the smallest amount of postulated basis. The great problem of squaring the circle, now with good reason held in low esteem, was the struggle of centuries to bring under the dominion of the prescribed means what might with the utmost ease have been conquered by a very small additional allowance The attempt was unsuccessful; so was that of COLUMBUS to discover India from the west. But COLUMBUS commenced the addition of America to the known world; and in like manner the squarers of the circle, and their refuters, added field on field to the extent of geometry, and aided largely in the preparation for the modern form of mathematics. Very few of these additions would have been made, at or near the time when they were made, if it had satisfied the Greek mind to meet each difficulty, as it occurred, by permission to use additional assumptions in geometry.

"The remains of the Hindoo algebra and geometry show to us no vestige of any attempt to gain force of thought by struggling against limitation of means: this, of course, because their mode of demonstration does not appear in the works which are left, or at least in those which have become known to Englishmen. But we have here a native of India who turns aside, at no suggestion but that of his own mind, and applies himself to a problem which has hitherto been assigned to the differential calculus, under the condition that none but purely algebraic process shall be used. He did not learn this course of proceeding from his European guides, whose aim it has long been to push their readers into the differential calculus with injurious speed, that they may reach the full application of mathematics to physics; and who often allow their pupils to read EUCLID with eyes shut to his limitations. RAMCHUNDRA proposed to himself a problem which a beginner in the differential calculus masters with a few strokes of the pen in a month's study, but which might have been thought hardly within the possibilities of pure algebra. His victory over the theory of the difficulty is complete. Many mathematicians of sufficient power to have done as much would have told him, when he first began, that the end proposed was perhaps unattainable by any amount of thought; next, that when attained, it would be of no use. But he found in the demands of his own spirit an impulse towards speculation of a character more fitted to the state of his own community than the imported science of his teachers. He applied to the branch of mathematics which is indigenous in India, the mode of thought under which science made its greatest advances in Greece. My own strong suspicion that it was the want of this mode of thought which allowed the decline of algebra in ancient India, coupled with my thorough conviction that, whether or no, this mode of thought yields the proper nutriment for mathematical science in its early and feeble life, produced the recommendation to the Court of Directors to which this reprint owes its own existence.

"RAMCHUNDRA's problem — and I think it ought to go by that name, for I cannot find that it was ever current* as an exercise of ingenuity in Europe — is to find the value of a variable which will make an algebraical function a maximum or a minimum, under the following condi-

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* It would not at all surprise me if it should be found that some one inquirer has suggested the problem; but, if so, I think the search which I have made entitles me to say that the suggestion entirely failed to attract attention, and to establish the difficulty as a recognized exercise.

tions. Not only is the differential calculus to be excluded, but even that germ of it which, as given by FERMAT in his treatment of this very problem, made some think that he was entitled to claim the invention. The values of ϕx and of $\phi\ (x + h)$ are not to be compared; and no process is to be allowed which immediately points out the relation of ϕx to the derived function $\phi' x$. A mathematician to whom I stated the conditioned problem made it, very naturally, his first remark, that he could not see how on earth I was to find out when it would be biggest, if I would not let it grow. The mathematician will at last see that the question resolves itself into the following: — Required a constant, r, such that $\phi x - r$ shall have a pair of equal roots, without assuming the development of $\phi\ (x + h)$ or any of its consequences.

"RAMCHUNDRA, the author of this work, has transmitted to me some notes of his own life, from which I collect, as follows. He was born in 1821, at Paneeput, about fifty miles from Delhi. His father, SOONDUR LALL, was a Hindoo Kaeth, and a native of Delhi, and was there employed under the collector of the revenue. He died at Delhi, in 1831-2, leaving a widow (who still survives) and six sons. After some education in private schools, RAMCHUNDRA entered the English Government school at Delhi, to every pupil of which two rupees a month were given, and a scholarship of five rupees a month to all in the first and second classes. In this school he remained six years. It does not appear that any particular attention was paid to mathematics in this school; but, shortly before leaving it, a taste for that science developed itself in RAMCHUNDRA, who studied at home with such books as he could procure. After leaving school, he obtained employment as a writer for two or three years. In 1841, changes took place in the educational department of the Bengal presidency; the school was formed into a college; and RAMCHUNDRA obtained, by competition, a senior scholarship, with thirty rupees a month. In 1844, he was appointed teacher of European science in the Oriental department of the college, through the medium of the vernacular, with fifty rupees a month additional. A vernacular translation society was instituted, and RAMCHUNDRA, in aid of its object, translated or compiled works in Oordoo, and also on algebra, trigonometry, &c., up to the differential and integral calculus."

A Treatise on Differential Equations. By GEORGE BOOLE, F. R. S., Professor of Mathematics in the Queen's University, Ireland, Honorary Member of the Cambridge Philosophical Society. 485 pp. Crown, 8vo. Cambridge: MACMILLAN & CO., and 23 Henrietta Street, Covent Garden, London. 1859.

The works devoted exclusively to a systematic treatment of Differential Equations, are neither so many, nor so exhaustive, that teachers and students will not welcome another, provided it will be well adapted to the wants of elementary instruction. To give the author's idea of these wants, and how they are to be supplied, we give the following extract from the Preface.

"It was my object first of all, to meet the wants of those who had no previous acquaintance with the subject, but I also desired not quite to disappoint others who might seek for more advanced information. These distinct, but not inconsistent aims determined the plan of composition. The earlier sections of each chapter contain that kind of matter which has usually been thought suitable for the beginner, while the latter ones are devoted either to an account of recent discovery, or to the discussion of such deeper questions of principle as are likely to present themselves to the reflective student in connection with the methods and processes of his previous course. The principles which I have kept in view in carrying out the above design are the following:

"1st. In the exposition of methods I have adhered as closely as possible to the historical order of their development. I presume that few who have paid any attention to the history of the Mathematical Analysis, will doubt that it has been developed in a certain order, or that that order has been, to a great extent, necessary — being determined, either by steps of logical de-

duction, or by the successive introduction of new ideas and conceptions, when the time for their evolution had arrived. And these are causes which operate in perfect harmony. Each new scientific conception gives occasion to new applications of deductive reasoning; but those applications may be only possible through the methods and the processes which belong to an earlier stage.

"Thus, to take an illustration from the subject of the following work: The solution of ordinary simultaneous differential equations properly precedes that of linear partial differential equations of the first order; and this, again, properly precedes that of partial differential equations of the first order which are not linear. And in this natural order were these subjects developed. Again, there exists large and very important classes of differential equations, the solution of which depends on some process of successive reduction. Now such seems to have been effected, at first, by a repeated change of variables; afterwards, and with greater generality, by a combination of such transformations with others involving differentiation; last of all, and with greatest generality, by symbolical methods. I think it necessary to direct attention to instances like these, because the indications which they afford, appear to me to have been, in some work, of great ability, overlooked; and because I wish to explain my motives for departing from the precedent thus set.

"Now there is this reason for grounding the order of exposition upon the historical sequence of discovery, that by so doing we are most likely to present each new form of truth to the minds precisely at that stage at which the mind is most fitted to receive it, or even like that of the discoverer, to go forth to meet it."

The plan of composition indicated in the above extract is completely carried out in nearly every chapter, of which there are eighteen. The elementary articles of each chapter, which the beginner should first read, are indicated by the author. A valuable feature of the work is the collection of interesting examples appended to each chapter; in all, over 300. At the end of the volume are found the answers of nearly one half the problems, which will be of great convenience to the student while acquiring confidence in his knowledge of the theory. Besides a general survey of the whole volume, we have read several of the earlier chapters with care, to be able to judge of its adaptation to the wants of students in this country; and we can commend it as being the very best work we have ever seen for a text-book upon the subject of Differential Equations; nor need the mathematician fear that he will fail to find it a quite full compendium of our present knowledge of this important department of analysis.

Editorial Items.

THE following gentlemen have sent us solutions of the Prize Problems in the August number of the MONTHLY: —

DAVID TROWBRIDGE, Perry City, Schuyler Co., N. Y., answered all the questions. His solutions of the Prize Problems in the July number did not reach us in time; as was also the case with the solutions sent by JAMES M. INGALS, of Delton, Sauk Co., Wis.

ASHER B. EVANS, Madison University, Hamilton, N. Y., answered all the questions.

GUSTAVUS FRANKENSTEIN, Springfield, Ohio, answered all the questions.

James F. Roberson, Senior Class, Indiana University, Bloomington, answered question III.

We have also received a set of excellent solutions from Asaph Hall, Esq., Assistant at Harvard College Observatory, who does not wish to compete for the prizes.

Prof. Werden Reynolds, late Principal of the Worcester Academy, has accepted the Presidency of the Worcester Female College. . . . . Prof. B. S. Hedrick has been appointed to the Professorship of Mathematics in Cooper Institute, N. Y. . . . . Rev. Thomas Hill will deliver a course of lectures before the Lowell Institute, in Boston, in a few weeks, and return to Yellow Springs, Ohio, about January 1, 1860. . . . . Prof. Chauvenet, of the U. S. Naval Academy at Annapolis, Md., has accepted a Professorship in Washington University, at St. Louis, Mo. . . . . William Watson, Esq., a graduate of and Tutor in the Lawrence Scientific School, already well and favorably known to our readers, will sail for Paris on the 12th of November, where he proposes to spend a year or more in the study of the Mathematics. Mr. Watson has kindly consented to be an occasional correspondent of the Mathematical Monthly during his absence, and send us such information concerning the schools and methods of instruction, especially in the Mathematics, as it may be in his power to give. Instead of the hasty observations of the mere tourist, we shall be informed of every day school life abroad, by one whose education and experience as a teacher of the mathematics will enable him to judge of those things which will most benefit his fellow teachers at home.— Our best wishes go with him. . . . . It gives us pleasure to add the following names to our list of coöperators and contributors: E. A. Strong, Esq., Grand Rapids, Mich.; Henry Ward Poole, Esq., Boston; Lucius Brown, Esq., Fall River, Mass.; Saxe Gotha Laws, Esq., Dover, Delaware; Chauncey Smith, Esq., Boston.

The portrait of Dr. Bowditch, which we have the gratification of presenting to our readers, needs no commendation at our hands, as a work of art; and we will also add that it is recognized by his family as a faithful likeness. It is eminently proper that the first portrait given in the Monthly should be of the "Father of American Geometry." We commend the list of his writings, with the interesting and valuable series of notes by his son, N. I. Bowditch, Esq., to the attention of all. We trust it may not be long before we shall have the pleasure of presenting the portraits of other eminent mathematicians, both of our own and foreign countries. . . . . Books Received. Tables of Victoria, Computed with regard to the Perturbations of Jupiter and Saturn. By F. Brünnow, Ph. Dr., Professor of Astronomy in the University of Michigan, and Director of the Observatory at Ann Arbor. Printed by order of the Board of Regents. New York: B. Westermann & Co. London: Trübner & Co. 4to. pp. 75. 1859. . . . . Linear Perspective Explained. By William N. Bartholomew, Author of Bartholomew's Sketch Book, and Series of Drawing Books, in six numbers. Boston: Shepard, Clark & Brown. 1859. . . . . Résumé de Leçons de Géométrie Analytique et de Calcul Infinitésimal. By J. B. Belanger, Ingénieur en chef des Ponts et Chaussées, Professeur de Mécanique à l'Ecole Impériale Polytechnique et à l'Ecole Centrale des Arts et Manufactures. Seconde Edition. Paris: Mallet-Bachelier. 1859. . . . . Journal de Mathématiques Pures et Appliquées; Publié by Joseph Liouville. Juillet. 1859. . . . . Nouvelles Annales de Mathématique Rédigé Par MM. Terquem et Gerono. Septembre, 1859.

THE

MATHEMATICAL MONTHLY.

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PRIZE PROBLEMS FOR STUDENTS.

I. If two circles touch each other, any straight line passing through the point of contact cuts off similar parts of their circumferences.

II. Find the four roots of the recurring equation

$$x^4 - \tfrac{5}{2}x^3 + 2x^2 - \tfrac{5}{2}x + 1 = 0.$$

III. If $2\cos\theta = u + \frac{1}{u}$, prove that $2\cos 2\theta = u^2 + \frac{1}{u^2}$, $2\cos 3\theta = u^3 + \frac{1}{u^3} \ldots\ldots 2\cos n\theta = u^n + \frac{1}{u^n}$; and then find the sum of the series, $\cos\theta + \cos 2\theta + \cos 3\theta \ldots\ldots + \cos n\theta$.

IV. Having given the Right Ascensions and Declinations of two stars, to find the formula for the distance between them. Also, find what the distance becomes, when for one star A. R. is $8^{h}\ 12^{m}\ 38^{s}.17$, and Dec. $17°\ 23'\ 49''.8$ north, and for the other A. R. is $13^{h}\ 28^{m}\ 19^{s}.92$, and Dec. $21°\ 12'\ 37''.2$ south.

V. In a frustum of any pyramid or cone, the area of a section, parallel to the two bases and equidistant from them, is the arithmetical mean of the arithmetical and geometrical means of the areas of the two bases.

The solutions of these problems must be received by February 1, 1860.

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. XI. Vol. I.

The first Prize is awarded to Gustavus Frankenstein, Springfield, Ohio.

The second Prize is awarded to Asher B. Evans, Madison University, Hamilton, New York.

Prize Solution of Problem I.

By Gustavus Frankenstein, Springfield, Ohio.

Solve the equations

$$x + y = a$$
$$(x^3 + y^3)(x^2 + y^2) = b,$$

and give a discussion of the values of the roots.

Squaring and cubing $x + y = a$, we get

$$x^2 + 2xy + y^2 = a^2,\ x^3 + 3x^2y + 3xy^2 + y^3 = a^3.$$

$$\therefore x^2 + y^2 = a^2 - 2xy,\ x^3 + y^3 = a^3 - 3xy(x + y) = a^3 - 3axy.$$

$$\therefore (x^3 + y^3)(x^2 + y^2) = (a^3 - 3axy)(a^2 - 2xy) = b.$$

$$\therefore 6ax^2y^2 - 5a^3xy + a^5 = b;$$

and solving we get

$$xy = \frac{5a^3 \pm \sqrt{a(24b + a^5)}}{12a} = q.$$

Hence, knowing the sum, a, of x and y, and their product, q, their values will be given by the quadratic $x^2 - ax + q = 0$.
Solving we get

$$x = \frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 - 4q} = \frac{a}{2} \pm \frac{1}{2}\sqrt{-\frac{1}{6}a^2 \mp \frac{\sqrt{a(24b + a^5)}}{12a}}.$$

Discussion. — *Case* I. *When a and b have the same signs.* Since $-\frac{1}{6}a^2$ is negative, whether a be positive or negative, it is evident that two of these values of x will always be imaginary. If a and b are of the same sign, q is real, since $a(24b + a^5)$ is positive; and x

will have two real values when $\sqrt{a(24b + a^5)} > 2a^3$; or when $a(24b + a^5) > 4a^6$; or $8b > a^5$. If this condition be not fulfilled, all the roots are imaginary. When $8b = a^5$ the roots are equal.

Case II. *When a and b have different signs.* In this case $\sqrt{a(24b + a^5)}$ will be imaginary unless $24b < a^5$, after which, that x may be real, $8b$ should be $=$ or $> a^5$; but as these conditions are contradictory, all the roots in this case are imaginary. Hence the conditions that there may be two real roots are, first, that a and b must have the same sign; and second, that $8b =$ or $> a^5$ numerically.

Indeed, by referring to the given equations, we see that if x and y are real, $x^2 + y^2$ is positive; and since $x^3 + y^3$ is always of the same sign as $x + y = a$, therefore b will also be of the same sign as a. If we suppose the roots equal, the proposed equations become $2x = a$, and $4x^5 = b$. Therefore, $4\left(\frac{a}{2}\right)^5 = b$; or $8b = a^5$; which, as before, exhibits the transition from real to imaginary roots.

If $\frac{1}{2}a + z$ and $\frac{1}{2}a - z$ be substituted for x and y in the proposed equations, the resulting equation in z will be

$$48az^4 + 16a^3z^2 + a^5 - 8b = 0.$$

Now a being real, z must be real, for all real values of $x = \frac{1}{2}a \pm z$; therefore, for real values of x, z^4 and z^2 are essentially positive. But in this equation the term containing z^3 is wanting, and since the signs of its adjacent terms are the same, whether a be positive or negative, it appears from DESCARTES's theory of signs that z has always two imaginary roots; and when the signs of a and b are unlike, all the terms will be either positive or negative for real values of z, and therefore their sum cannot be nothing; hence, $x = \frac{1}{2}a \pm z$ cannot be real when a and b have different signs. But when a and b have like signs, there will be two real roots when $8b > a^5$, for then the equation, having its last term negative, cannot have all its roots imaginary. When $8b = a^5$, $z^2 = 0$, which corre-

sponds to the equal roots of $x = \frac{1}{2} a \pm 0 = y$, the other factor giving the imaginary roots.

Prize Solution of Problem II.

By Gustavus Frankenstein, Springfield, Ohio.

Let A, B, C be the angles, and a, b, c the opposite sides, of a plane triangle; it is required from the relation

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

to deduce the formula

$$a^2 = b^2 + c^2 - 2\, b\, c \cos A.$$

Since $A + B + C = 180°$, $C = 180° - (A + B)$; and

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

gives

$$(1.) \qquad \begin{aligned} c \sin A = a \sin C &= a \sin(A + B) \\ &= a\,(\sin A \cos B + \cos A \sin B). \end{aligned}$$

Substitute for $\sin B$ its value $\frac{b \sin A}{a}$, and for $\cos B$ its value $\frac{\sqrt{a^2 - b^2 \sin^2 A}}{a}$, and (1) becomes, after dividing by $\sin A$,

$$c = \sqrt{a^2 - b^2 \sin^2 A} + b \cos A.$$

$$\therefore a^2 - b^2 \sin^2 A = (c - b \cos A)^2 = c^2 - 2\, b\, c \cos A + b^2 \cos^2 A;$$

$$\begin{aligned} \therefore a^2 &= c^2 + b^2 (\sin^2 A + \cos^2 A) - 2\, b\, c \cos A \\ &= c^2 + b^2 - 2\, b\, c \cos A. \end{aligned}$$

Prize Solution of Problem III.

By Asaph Hall, Assistant at Harvard College Observatory.

A number n of equal circles touch each other externally, and include an area of a square feet; to find the radii of the circles. — Communicated by Artemas Martin, Esq.

Joining the centres of the equal circles, we shall have a regular polygon of n sides, each side being equal to $2\, r$, twice the required radius.

Join the centre of the polygon with the centre of each circle. In each of the n equal triangles thus formed, the angle at the centre of the polygon is one nth of four right angles, or $\frac{2\pi}{n}$; and the sum of the remaining angles of each triangle is $\pi - \frac{2\pi}{n} = \frac{(n-2)\pi}{n} =$ the angle of each of the n equal sectors. The altitude of each triangle is $r \cot \frac{\pi}{n}$, and therefore $r^2 \cot \frac{\pi}{n}$ is the area of each triangle, and $n r^2 \cot \frac{\pi}{n}$ the area of the polygon. The area of each sector is $\frac{r^2 (n-2)\pi}{2n}$, and of the n sectors, $\frac{r^2 (n-2)\pi}{2}$.

$$\therefore n r^2 \cot \frac{\pi}{n} - \tfrac{1}{2} r^2 (n-2)\pi = a,$$

$$\therefore r = \sqrt{\frac{2a}{2n \cot \frac{\pi}{n} - (n-2)\pi}}.$$

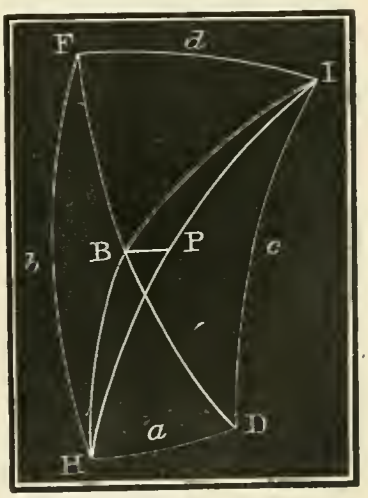
Prize Solution of Problem IV.

By David Trowbridge, Perry City, Schuyler Co., N. Y.

If the sides of a spherical trapezium be denoted by a, b, c, d, the diagonals by δ_1 and δ_2, and the distance between the middle points of the diagonals by $\varDelta$; show that

$\cos a + \cos b + \cos c + \cos d = 4 \cos \frac{1}{2} \delta_1 \cos \frac{1}{2} \delta_2 \cos \varDelta$.

— Communicated by George Eastwood, Esq.



Let B and P be the middle points of the diagonals δ_1 and δ_2, then in the triangles FBH, HBD, DBI, IBF, we have from a fundamental formula of spherical trigonometry

$$\cos b = \cos \tfrac{1}{2} \delta_1 \cos BH + \sin \tfrac{1}{2} \delta_1 \sin BH \cos HBF,$$
$$\cos a = \cos \tfrac{1}{2} \delta_1 \cos BH + \sin \tfrac{1}{2} \delta_1 \sin BH \cos HBD,$$
$$\cos c = \cos \tfrac{1}{2} \delta_1 \cos BI + \sin \tfrac{1}{2} \delta_1 \sin BI \cos DBI,$$
$$\cos d = \cos \tfrac{1}{2} \delta_1 \cos BI + \sin \tfrac{1}{2} \delta_1 \sin BI \cos IBF,$$
$$\therefore \cos a + \cos b + \cos c + \cos d = 2 \cos \tfrac{1}{2} \delta_1 (\cos BH + \cos BI),$$

since HBF and HBD, as well as DBI and IBF, are supplementary, their cosines having opposite signs. But in the triangles BIP and BHP we have

$$\cos BH = \cos \tfrac{1}{2}\,\delta_2 \cos \varDelta + \sin \tfrac{1}{2}\,\delta_2 \sin \varDelta \cos BPH,$$

$$\cos BI = \cos \tfrac{1}{2}\,\delta_2 \cos \varDelta + \sin \tfrac{1}{2}\,\delta_2 \sin \varDelta \cos BPI,$$

$\therefore \cos BH + \cos BI = 2 \cos \tfrac{1}{2}\,\delta_2 \cos \varDelta$, since the angles BPH and BPI are supplementary.

$$\therefore \cos a + \cos b + \cos c + \cos d = 2 \cos \tfrac{1}{2}\,\delta_1 (\cos BH + \cos BI)$$
$$= 4 \cos \tfrac{1}{2}\,\delta_1 \cos \tfrac{1}{2}\,\delta_2 \cos \varDelta.$$

Prize Solution of Problem V.

By Asher B. Evans, Madison University, Hamilton, N. Y.

From an urn containing four white and four black balls, four are repeatedly drawn and replaced. A agrees to pay B one dollar every time the four balls drawn are equally divided between white and black; but if three, or all four, are of the same color, B is to pay A one dollar. Who has the advantage, and what is its value for each drawing? — Communicated by Simon Newcomb, Esq.

The whole number of different fours which can be drawn out of the eight balls is

$$\overset{4}{\underset{8}{C}} = \frac{8\,(8-1)\,(8-2)\,(8-3)}{1\,.\,2\,.\,3\,.\,4} = 70.$$

Since "A agrees to pay B one dollar every time the four balls are equally divided between white and black," the number of the 70 in B's favor is

$$\overset{2}{\underset{4}{C}} \times \overset{2}{\underset{4}{C}} = 36,$$

and the number in A's favor, since three of the four may be white and one black, three black and one white, all four white, all four black, is

$$2\,(\overset{3}{\underset{4}{C}} \times \overset{1}{\underset{4}{C}} + \overset{4}{\underset{4}{C}}) = 34.$$

Therefore B has the advantage, which amounts to two dollars in 70 draws, or one thirty-fifth of a dollar for each drawing.

Chauncey Wright.
Truman Henry Safford.

NOTES AND QUERIES.

1. *The greatest Common Divisor.* *Case* I. *When the given numbers are such as may be readily factored.* The following process is based upon the well-known principle of dividing by component factors. It is evident that if several numbers have a common divisor, they may all be divided by any component factor of this divisor, and the resulting quotients by another of the component factors; and so on.

EXAMPLE.

2	84	126	210	252	294	462
3	42	63	105	126	147	231
7	14	21	35	42	49	77
	2	3	5	6	7	11

Greatest common divisor $= 2 \times 3 \times 7 = 42$.

Case II. *When the factors of the given numbers are not readily apparent.* The well-known rule of dividing the greater by the less, the last divisor by the last remainder, &c., need not be repeated. But the *method*, or *form of work*, given here, is recommended to teachers as being more concise and elegant than the usual method, requiring less time and less space on the slate or blackboard.

OPERATION.

		167463
73761	2	147522
59823	3	19941
13938	1	13938
12006	2	6003
1932	3	5796
1863	9	207
69	3	207

Ans. 69.

EXAMPLE.

What is the greatest common divisor of 73761 and 167463 ?

The operation needs no explanation. — J. C. PORTER, Professor of Mathematics in Clinton Liberal Institute, Clinton, N. Y.

2. *Equation of Payments.* — There is probably no mercantile calculation that is more tedious than the averaging of accounts, or Equation of Payments. The labor of computation may be much

diminished by performing all the multiplications at once, — first multiplying by all the units' figures of the several multipliers, then by all the tens' figures, and so on.

Example.

Required the average time of payment of \$ 371 due in 15 days, \$ 25 due in 84 days, \$ 1603 due in 107 days, and \$ 885 due in 138 days.

$$\begin{array}{r}
371 \times 15 \\
25 \times 84 \\
1603 \times 107 \\
885 \times 138 \\
\hline
20256 \\
3226 \\
2488 \\
\hline
301316
\end{array}$$

Commencing with the units' figures of the multipliers, say $8 \times 5 + 7 \times 3 + 4 \times 5 + 5 \times 1 = 86$. Set down 6, and carry 8. $8 + 8 \times 8 + 7 \times 0 + 4 \times 2 + 5 \times 7 = 115$. Set down 5, and carry 11. $11 + 8 \times 8 + 7 \times 6 + 5 \times 3 = 132$. Set down 2, and carry 13. $13 + 7 \times 1 = 20$, which is to be set down.

$3 \times 5 + 0 \times 3 + 8 \times 5 + 1 \times 1 = 56$. Set down 6 in tens' place, and carry 5. $5 + 3 \times 8 + 0 \times 0 + 8 \times 2 + 1 \times 7 = 52$. Set down 2, and carry 5. $5 + 3 \times 8 + 0 \times 6 + 1 \times 3 = 32$, which set down.

$1 \times 5 + 1 \times 3 = 8$, which is to be set down in hundreds' place. $1 \times 8 + 1 \times 0 = 8$, which set down. $1 \times 8 + 1 \times 6 = 14$. Set down 4, and carry 1. $1 + 1 \times 1 = 2$, which set down. Adding these partial products, we find the sum of the several products is 301316, as may be easily proved by multiplying in the usual way.

A little practice will enable one to perform the multiplications with nearly as much facility as simple addition. The *products* should be mentally announced and added together, thus: 40 and 21 and 20 and 5 are 86. 8 and 64 and 8 and 35 are 115. 11 and 64 and 42 and 15 are 132. 13 and 7 are 20, &c., &c. — Pliny Earle Chase, Philadelphia, Pa.

3. *Decomposition of the irreducible rational fraction* $\frac{f(x)}{((x-\alpha)+\beta^2)^n}$ *into simple fractions of the forms*

$$\frac{ax+b}{((x-\alpha)^2+\beta^2)^n},\ \frac{a'x+b'}{((x-\alpha)^2+\beta^2)^{n-1}},\ \&c.$$

Divide $f(x)$ by $(x-\alpha)^2+\beta^2$; the remainder of this division will be the numerator of the first fraction sought. For the division will give an equality of the form

$$f(x)=((x-\alpha)^2+\beta^2)\,f_1(x)+a\,x+b;$$

whence

$$\frac{f(x)}{((x-\alpha)^2+\beta^2)^n}=\frac{ax+b}{((x-\alpha)^2+\beta^2)^n}+\frac{f_1(x)}{((x-\alpha)^2+\beta^2)^{n-1}}.$$

By dividing in like manner the quotient $f_1(x)$ by $(x-\alpha)^2+\beta^2$, the remainder of this second division will be the numerator $a'x+b'$ of the second fraction sought; and so on. — *Nouvelles Annales de Mathématiques*, Septembre, 1859.

The student will see the simplicity of this method of decomposition from the following example. Decompose $\frac{5x^3+6x^2-8x+20}{(x-2)^4}$.

$$\frac{5x^3+6x^2-8x+20}{(x-2)^4}=\frac{68}{(x-2)^4}+\frac{5x^2+16x+24}{(x-2)^3}$$

$$=\frac{68}{(x-2)^4}+\frac{76}{(x-2)^3}+\frac{5x+26}{(x-2)^2}$$

$$=\frac{68}{(x-2)^4}+\frac{76}{(x-2)^3}+\frac{36}{(x-2)^2}+\frac{5}{x-2}.$$

It is obvious that this method can be applied to any fraction of the form $\frac{f(x)}{(F(x))^n}$, in which $f(x)$ is of a higher degree than $F(x)$.

REVIEW OF THE PRIZE SOLUTION OF THE LAST PROBLEM IN EMERSON'S NORTH AMERICAN ARITHMETIC.*

By Hon. Finley Bigger, Register U. S. Treasury, Washington, D. C.

The following review of the Prize Solution of Problem 137 in Emerson's North American Arithmetic, Part III., was submitted to the National Teachers' Association at its late session in Washington, and referred to the Mathematical Monthly for publication.

For the purpose of elucidation, it is assumed that the question is susceptible of two constructions. The one adopted in the Prize Solution considers each term of supposition, compared with the term of demand, as separate and distinct propositions, and that the words "the grass being at first equal on every acre, and growing uniformly," demand that the acres, in each condition of the question speci-

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* Prize Question 137. — "If 12 oxen eat up $3\frac{1}{2}$ acres of grass in 4 weeks, and 21 oxen eat up 10 acres in 9 weeks, how many oxen will eat up 24 acres in 18 weeks; the grass being, at first, equal on every acre, and growing uniformly?"

In June, 1835, a premium of $ 50 was offered for the most "lucid analytical solution" of the last question in the Third Part of Emerson's North American Arithmetic; and subsequently a committee to examine the solutions presented, and award the premium, was raised in the manner proposed. The committee have given a very careful and patient attention to the labors of the trust confided to them, and they now make the following

REPORT.

The whole number of solutions presented was 112, of which 48 gave the true answer. After excluding those solutions which gave incorrect answers, the committee proceeded to diminish the remaining number, by excluding those which were algebraical, and also those which were performed either by *position* or *proportion*, retaining for the comparative examination such only as were strictly analytical. The solution for which the committee have awarded the premium was presented by James Robinson, Principal of the Department of Arithmetic, Bowdoin School, Boston. It is as follows: —

Solution. — It is evident that a part of the given number of oxen, in each condition of this question, must be supported by the grass *at first standing* on the given number of acres, and that the remaining part must be supported by the *growth*. It is also evident that the number of oxen that can be supported by the grass at first standing on the ground must be in a direct ratio to the number of acres, and in an inverse ratio to the time of grazing. And it is further obvious that the number of oxen that can be supported by the growth of the grass must be in a direct ratio to the number of acres, without any regard to the *time* of grazing, because the

fied, shall be so increased proportionally, as that the answer to each supposition, thus separately considered, shall be precisely the same in amount, or such that one answer will be alike the ratio of either.

The other insists that the words "*being* equal and *growing* uniformly," are merely suggestive of condition, and do not authorize an increase of the numerical expression of the acres of grass specified in the terms of the proposition.

This second interpretation regards the alleged qualifying language as indicating no mathematical ratio, or measure of value, and concludes, therefore, that the solution of the question must proceed as if this phraseology were omitted, and the acres designated as tons of hay. And, thus considered, but one answer is possible, and this the expression of the mean ratio of the two terms of supposition through the one term of demand.

We offer a few words by way of analysis and criticism of the

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number of oxen that would consume the growth of any given number of acres during any given time, would consume the same growth continually.

By the first condition of the question, 12 oxen consume $3\frac{1}{2}$ acres of grass and its growth in 4 weeks; the 10 acres being $\frac{20}{7}$ of $3\frac{1}{2}$ acres, it would require $\frac{20}{7}$ as many oxen to consume 10 acres of grass and its growth in the same time; and 12 oxen multiplied by $\frac{20}{7}$ are $34\frac{2}{7}$ oxen. To consume the same in 9 weeks would require only $\frac{4}{9}$ as many oxen, and $34\frac{2}{7}$ oxen multiplied by $\frac{4}{9}$ are $15\frac{5}{21}$ oxen.

By the second condition, 21 oxen consume 10 acres of grass and its growth in 9 weeks, and 21 oxen less $15\frac{5}{21}$ oxen are $5\frac{16}{21}$ oxen. Then it follows, that $5\frac{16}{21}$ oxen in 9 weeks would consume the growth of 10 acres of grass during the 5 remaining weeks. To consume the growth of 10 acres during 9 weeks would require $\frac{9}{5}$ as many oxen, and $5\frac{16}{21}$ oxen multiplied by $\frac{9}{5}$ are $10\frac{13}{35}$ oxen. Then, 21 oxen less $10\frac{13}{35}$ oxen are $10\frac{22}{35}$ oxen. Hence it is evident that $10\frac{22}{35}$ oxen, in 9 weeks, would consume the grass at first on the 10 acres; and it is also evident that $10\frac{13}{35}$ oxen, in 9 weeks, would consume the growth of the 10 acres of grass during the 9 weeks.

The 24 acres in the third condition being $\frac{24}{10}$, or $2\frac{2}{5}$ times 10 acres, it would require $2\frac{2}{5}$ times $10\frac{22}{35}$ oxen to consume the grass at first on the 24 acres in 9 weeks; and $10\frac{22}{35}$ oxen multiplied by $2\frac{2}{5}$ are $25\frac{89}{175}$ oxen. To consume the same in 18 weeks would require only $\frac{9}{18}$, or $\frac{1}{2}$ as many oxen; and $25\frac{89}{175}$ oxen, divided by 2, are $12\frac{132}{175}$ oxen. And to consume the growth of the 24 acres of grass during the 18 weeks would require $2\frac{2}{5}$ times $10\frac{13}{35}$ oxen; and $10\frac{13}{35}$ oxen multiplied by $2\frac{2}{5}$ are $24\frac{156}{175}$ oxen.

Lastly, $12\frac{132}{175}$ oxen plus $24\frac{156}{175}$ oxen are $37\frac{113}{175}$ oxen, the number required.

By order of the Committee,

P. MACKINTOSH, *Chairman*.

Prize Solution. By analysis the author arrived at the following results. "Hence, it is evident that $10\frac{22}{35}$ oxen, in 9 weeks, would consume the grass at first standing on the 10 acres; and it is also evident that $10\frac{13}{35}$ oxen, in 9 weeks, would consume the growth of 10 acres during the 9 weeks." If 10 acres, in 9 weeks, grow 10 acres, 24 acres in 18 weeks would grow 48 acres; and thus the Prize Solution presents us with the following propositions, which produce the final result. If $10\frac{22}{35}$ oxen, in 9 weeks, eat the grass at first on 10 acres, how many oxen will eat the grass at first on the 24 acres, in 18 weeks? and if $10\frac{13}{35}$ oxen, in 9 weeks, eat the growth of 10 acres, how many oxen will eat the growth of 24 acres, which is 48, in 18 weeks? $\frac{24 \times 9 \times 10\frac{22}{35}}{10 \times 18} = 12\frac{132}{175}$ oxen; $\frac{48 \times 9 \times 10\frac{13}{35}}{10 \times 18} = 24\frac{156}{175}$ oxen, and $12\frac{132}{175} + 24\frac{156}{175} = 37\frac{113}{175}$ oxen, the same as given by the Prize Solution. If this be what it purports, the true answer, by inverting the oxen in the above statement, the same answer *must* be obtained.

Thus $\frac{24 \times 9 \times 10\frac{13}{35}}{10 \times 18} = 12\frac{78}{175}$ oxen; $\frac{48 \times 9 \times 10\frac{22}{35}}{10 \times 18} = 25\frac{89}{175}$ oxen, and $12\frac{78}{175} + 25\frac{89}{175} = 37\frac{167}{175}$ oxen, the answer. And thus is exhibited the fact that this solution, as well as $37\frac{113}{175}$, fulfils the conditions of the question; for $37\frac{167}{175}$ oxen is equally the answer with $37\frac{113}{175}$ oxen.

The second interpretation furnishes the following solution. If 12 oxen, in 4 weeks, eat $3\frac{1}{2}$ acres of grass, how many oxen will eat 24 acres in 18 weeks? And, if 21 oxen, in 9 weeks, eat 10 acres of grass, how many oxen will eat 24 acres in 18 weeks? First $\frac{24 \times 12 \times 4}{3\frac{1}{2} \times 18} = 18\frac{2}{7}$ oxen; second, $\frac{24 \times 21 \times 9}{10 \times 18} = 25\frac{1}{5}$ oxen, and $18\frac{2}{7} + 25\frac{1}{5} = 43\frac{17}{35}$ oxen; $24 + 24 = 48$ acres. If then, $43\frac{17}{35}$ oxen, in 18 weeks, eat 48 acres of grass, it is self-evident that it will require $\frac{1}{2}$ of $43\frac{17}{35}$ oxen to eat $\frac{1}{2}$ the number of acres, 24, in

the same length of time, 18 weeks, and $\frac{1}{2}$ of $43\frac{17}{35} = 21\frac{26}{35}$ oxen, the answer.

NOTE. — We have appended the Prize Solution, to give our readers an opportunity to judge for themselves. It does not seem to us that there is any ambiguity in the statement of the problem, nor much difficulty in its solution; and we add the following for those who may wish to study it. Let *unity* denote the amount of grass at first on each acre, g the growth on each acre per week, and x the required number of oxen. Then, from the first condition, $1 + 4\,g$ is the amount of grass consumed from one acre in 4 weeks, $(1 + 4\,g)\,3\frac{1}{2}$ the amount from $3\frac{1}{2}$ acres, $\frac{(1 + 4\,g)\,3\frac{1}{2}}{12}$ the amount consumed by one ox in 4 weeks, $\frac{(1 + 4\,g)\,3\frac{1}{2}}{12 \times 4}$ the amount consumed by one ox in one week. From the second condition, $1 + 9\,g$ is the amount of grass consumed from one acre in 9 weeks, $(1 + 9\,g)\,10$ the amount from 10 acres, $\frac{(1 + 9\,g)\,10}{21}$ the amount consumed by one ox in 9 weeks, $\frac{(1 + 9\,g)\,10}{21 \times 9}$ the amount consumed by one ox in 1 week. From the third condition, $\frac{(1 + 18\,g)\,24}{x \times 18}$ is the amount consumed by one ox in 1 week. Therefore

$$\frac{(1 + 4\,g)\,3\frac{1}{2}}{12 \times 4} = \frac{(1 + 9\,g)\,10}{21 \times 9}$$

$$\frac{(1 + 18\,g)\,24}{x \times 18} = \frac{(1 + 9\,g)\,10}{21 \times 9}.$$

By solving these equations we get $x = 37\frac{113}{175}$, which is the only number of oxen which will satisfy all the conditions of the problem.

SOLUTION OF CUBIC EQUATIONS BY THE COMMON LOGARITHMIC TABLES.*

BY A CORRESPONDENT.

BY removing the second term, in the usual mode, every cubic takes the form $x^3 + a\,x = b$. Assume $x = y + z$, and $3\,y\,z = -a$. By substituting these, the given equation will become $y^3 + z^3 = b$. From the last two equations, the value of y^3 and of z^3 can be found by a quadratic. Since $x = y + z$, we thus obtain CARDAN's formula,

* For the trigonometrical solution of equations of the second, third, and fourth degrees, the student may consult CAGNOLI's *Trigonométrie*, Chap. XIV.; or CHAUVENET's *Trigonometry*, pp. 95 – 100, for the solution of equations of the second and third degrees. — ED.

$$x = \left(\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{\frac{1}{3}} + \left(\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{\frac{1}{3}}, \text{ or}$$

$$= \left(\frac{b}{2}\right)^{\frac{1}{3}} \left\{ \left(1 + \sqrt{1 + \left(\frac{2}{b}\right)^2 \left(\frac{a}{3}\right)^3}\right)^{\frac{1}{3}} + \left(1 - \sqrt{\left(\frac{2}{b}\right)^2 \left(\frac{a}{3}\right)^3}\right)^{\frac{1}{3}} \right\}.$$

In adapting this expression to the logarithmic tables three cases are presented.

I. *When a is positive.* Make $\tan v = \frac{2}{b}\left(\frac{a}{3}\right)^{\frac{3}{2}}$, or $\frac{b}{2\cos v} = \left(\frac{a}{3}\right)^{\frac{3}{2}} \div \sin v$; then by trigonometry,

$$x = \left(\frac{b}{2}\right)^{\frac{1}{3}} \{(1 + \sqrt{1 + \tan^2 v})^{\frac{1}{3}} + (1 - \sqrt{1 + \tan^2 v})^{\frac{1}{3}}\},$$

$$= \left(\frac{b}{2}\right)^{\frac{1}{3}} \left\{ \left(1 + \frac{1}{\cos v}\right)^{\frac{1}{3}} + \left(1 - \frac{1}{\cos v}\right)^{\frac{1}{3}} \right\},$$

$$= \sqrt{\frac{a}{3}} \left\{ \left(\frac{1 + \cos v}{\sin v}\right)^{\frac{1}{3}} - \left(\frac{1 - \cos v}{\sin v}\right)^{\frac{1}{3}} \right\},$$

$$= \sqrt{\frac{a}{3}} \left\{ \left(\cot \frac{v}{2}\right)^{\frac{1}{3}} - \left(\tan \frac{v}{2}\right)^{\frac{1}{3}} \right\}.$$

This equation may be further simplified by making $\tan u = \left(\tan \frac{v}{2}\right)^{\frac{1}{3}}$; then

$$x = \sqrt{\frac{a}{3}}\,(\cot u - \tan u).$$

$$= 2\sqrt{\frac{a}{3}} \cot 2u.$$

In this case there is but one real root. When b is negative it will only change the signs of v and u.

II. *When a is negative and of such value that $\frac{2}{b}\left(\frac{a}{3}\right)^{\frac{3}{2}}$ is less than* 1.

Let $\sin v = \frac{2}{b}\left(\frac{a}{3}\right)^{\frac{3}{2}}$; then

$$x = \left(\frac{b}{2}\right)^{\frac{1}{3}} \{(1 + \cos v)^{\frac{1}{3}} + (1 - \cos v)^{\frac{1}{3}}\}$$

$$= (b)^{\frac{1}{3}} \left\{ \left(\cos \frac{v}{2}\right)^{\frac{2}{3}} + \left(\sin \frac{v}{2}\right)^{\frac{2}{3}} \right\}.$$

For logarithmic computation, let

$$\tan u = \left(\tan\frac{v}{2}\right)^{\frac{1}{3}}, \text{ or } \frac{\cos^{\frac{2}{3}}\frac{v}{2}}{\cos^2 u} = \frac{\sin^{\frac{2}{3}}\frac{v}{2}}{\sin^2 u} = \frac{\left(\cos\frac{v}{2}\sin\frac{v}{2}\right)^{\frac{1}{3}}}{\cos u \sin u}.$$

$$x = b^{\frac{1}{3}} \cos^{\frac{2}{3}}\frac{v}{2}\,(1 + \tan^2 u) = \frac{b^{\frac{1}{3}}\cos^{\frac{2}{3}}\frac{v}{2}}{\cos^2 u}$$

$$= \frac{b^{\frac{1}{3}}\left(\frac{1}{2}\right)^{\frac{1}{3}}\sin^{\frac{1}{3}} v}{\frac{1}{2}\sin 2u} = \frac{2\sqrt{\frac{a}{3}}}{\sin 2u}.$$

In this case, also, there is but one real root; and when b is negative, the arcs v and u will be negative, as before.

III. "*The irreducible case.*" *When a is negative and of such value that $\frac{2}{b}\left(\frac{a}{3}\right)^{\frac{3}{2}}$ is greater than* 1.

Make $\cos v = \frac{b}{2}\left(\frac{3}{a}\right)^{\frac{3}{2}}$; then

$$x = \sqrt{\frac{a}{3}}\,\{(\cos v + \sqrt{-1}\sin v)^{\frac{1}{3}} + (\cos v - \sqrt{-1}\sin v)^{\frac{1}{3}}\}.$$

By applying Demoivre's theorem, the imaginary quantities disappear, leaving $x = 2\sqrt{\frac{a}{3}}\cos\frac{v}{3}$. But this $\cos\frac{v}{3}$ corresponds to the arcs v, $360° + v$, and $360° - v$. Dividing by 3, and putting 120° under the form of 180° — 60°, we find the other two roots,

$$x = -2\sqrt{\frac{a}{3}}\cos\left(60° - \frac{v}{3}\right),$$

$$x = -2\sqrt{\frac{a}{3}}\cos\left(60° + \frac{v}{3}\right).$$

We are now prepared to recapitulate; it being recollected that 10 should be algebraically subtracted from the index of the logarithmic sines and tangents. First bring the given cubic to the general form $x^3 + ax = b$.

I. When the coefficient a is positive. Find

$$\tan v = \frac{2}{b}\left(\frac{a}{3}\right)^{\frac{3}{2}};\ \tan u = \left(\tan\frac{v}{2}\right)^{\frac{1}{3}};\ x = 2\sqrt{\frac{a}{3}}\cot 2u.$$

II. When a is negative, and $\frac{2}{b}\left(\frac{a}{3}\right)^{\frac{3}{2}}$ less than 1. Find

$$\sin v = \frac{2}{b}\left(\frac{a}{3}\right)^{\frac{3}{2}};\ \tan u = \left(\tan\frac{v}{2}\right)^{\frac{1}{3}};\ x = \frac{2\sqrt{\frac{a}{3}}}{\sin 2u}.$$

III. When a is negative, and $\frac{2}{b}\left(\frac{a}{3}\right)^{\frac{3}{2}}$ greater than 1. Find

$$\cos v = \frac{b}{2}\left(\frac{3}{a}\right)^{\frac{3}{2}};\ \ x = 2\sqrt{\frac{a}{3}}\cos\frac{v}{3};\ \ x = -2\sqrt{\frac{a}{3}}\cos\left(60^\circ - \frac{v}{3}\right);\ \text{and}$$

$$x = -2\sqrt{\frac{a}{3}}\cos\left(60^\circ + \frac{v}{2}\right).$$

In the case where $\frac{2}{b}\left(\frac{a}{3}\right)^{\frac{3}{2}} = 1$, $x = 2\left(\frac{b}{2}\right)^{\frac{1}{3}}$.

Thus all the real roots of any cubic equation may be found by logarithms. It is perhaps unnecessary to remark, that in these values of x, the coefficient a is to be taken as numerically positive, irrespective of its algebraic sign. The investigation of these solutions is new in part, and will be found convenient for reference.

EQUATIONS OF THE SECOND DEGREE.

(1.) $$x^2 + px = q.$$

SOLUTION.

$$\tan A = \frac{2}{p}\sqrt{q},$$
$$x = \sqrt{q}\tan\tfrac{1}{2}A,$$
$$x = -\sqrt{q}\cot\tfrac{1}{2}A,$$

(2.) $$x^2 - px = q.$$

SOLUTION.

$$\tan A = \frac{2}{p}\sqrt{q},$$
$$x = -\sqrt{q}\tan\tfrac{1}{2}A,$$
$$x = \sqrt{q}\cot\tfrac{1}{2}A.$$

(3.) $$x^2 + px = -q.$$

(4.) $$x^2 - px = -q.$$

If $p^2 < 4q$ the roots of (3) and (4) are imaginary.

SOLUTION (3.).

$$\sin A = \frac{2}{p}\sqrt{q},$$
$$x = -\sqrt{q}\tan\tfrac{1}{2}A,$$
$$x = -\sqrt{q}\cot\tfrac{1}{2}A.$$

SOLUTION (4.).

$$\sin A = \frac{2}{p}\sqrt{q},$$
$$x = \sqrt{q}\tan\tfrac{1}{2}A,$$
$$x = \sqrt{q}\cot\tfrac{1}{2}A.$$

We have taken the liberty to add trigonometrical solutions of equations of the second degree, and commend them, as well as Correspondent's cubics, to the attention of students. — ED.

THE MOTIONS OF FLUIDS AND SOLIDS RELATIVE TO THE EARTH'S SURFACE.

[Continued from page 406, Vol· I.]

SECTION V.

ON THE MOTIONS OF THE ATMOSPHERE ARISING FROM LOCAL DISTURBANCES.

58. BESIDES the general disturbance of equilibrium arising from a difference of specific gravity between the equator and the poles, which causes the general motions of the atmosphere, treated in the last section, there are also more local disturbances, arising from a greater rarefaction of the atmosphere over limited portions of the earth's surface, which give rise to the various irregularities in its motions, including cyclones or revolving storms, tornadoes, and water-spouts. When, on account of greater heat, or a greater amount of aqueous vapor, the atmosphere at any place becomes more rare than the surrounding portions, it ascends, and the surrounding heavier atmosphere flows in below, to supply its place, while a counter current is consequently produced above. As the lower strata of atmosphere generally contain a certain quantity of aqueous vapor, which is condensed after arising to a certain height, and forms clouds and rain, the caloric given out in the condensation, in accordance with ESPY'S theory, produces a still greater rarefaction, and doubtless adds very much to the disturbance of equilibrium, and to the motive power of storms. So long, then, as the ascending atmosphere over the area of greater rarefaction is supplied with aqueous vapor by the current flowing in from all sides below, the disturbance of equilibrium must continue, and consequently the local disturbances of the atmosphere to which it gives rise, whether those of an ordinary rain storm, or a cyclone, may continue many days, while the general motions of the atmosphere may carry this disturbed area several thousands of miles.

59. In the ordinary rain-storms of the United States, the area of greater rarefication seems to be, in general, very oblong in the direction of the meridians, as is shown by Espy's charts. The atmosphere becoming more rare over the land, a current seems to set in from the Atlantic towards the Rocky Mountains, causing an ascent of the atmosphere in the west, and a line of greatest rarefaction in the direction of the meridians, arising from the condensation of the ascending vapor into clouds and rain, while the general motion of the atmosphere eastward, in those latitudes, carries this area of greater rarefaction, with its accompanying rain-storm, towards the east, at an average rate of about 30 miles per hour. As the velocity of the general eastward motion of the atmosphere is greater above, the rainy portion of the storm is for the most part on the east side of the line of greatest rarefaction, and as the currents below must be towards this line on both sides, when it passes over any place, the rain generally ceases and the wind changes.

60. When the area of rarefaction is such as to cause the atmosphere to flow in from all sides below towards a centre, and the reverse above, the disturbed portion of atmosphere, if it were not that its motions are resisted by the earth's surface, and the surrounding undisturbed part, would assume the outline and the gyratory motion in the case of no resistance, as represented in Fig. 3 and Fig. 4. But on account of the resistances, the motions of the atmosphere are very much modified, so that it has only a tendency to assume in some measure those motions, and instead of the atmosphere's receding entirely from the centre, on account of the rapidity of the gyrations near the centre, as represented in Fig. 3, it is only a little depressed in the middle, as represented in Fig. 6.

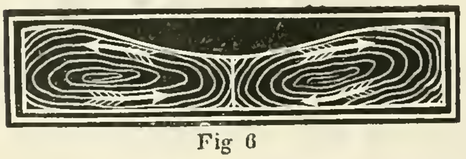

Fig 6

61. Since the force which produces the gyrations depends upon $D_t r$, that is, upon the velocity of the flow to and from the centre,

it is evident, that, at the centre and at the external part of the disturbed portion of atmosphere, where $D_t r$ must vanish, the resistances destroy all gyratory motion. Hence, instead of very rapid gyrations near the centre, as in the case of no resistances, there must be a calm there, and the most rapid gyrations be at some distance from the centre, in accordance with observation. The diameter of the comparatively calm portion, in the centre of the large cyclones, is sometimes about 30 miles. The velocity of gyration of the external part, which, in the case of no resistances, is small, is in a great measure destroyed by the resistances of the surrounding atmosphere, so that it is, for the most part, insensible to observation, and only the more rapid gyrations of the internal part are observed. The motion of gyration combined with the motion at the earth's surface towards the centre, gives rise to a spiral motion towards the centre, exactly in accordance with the observed motions of the atmosphere in great storms or hurricanes, as has been shown by Redfield, in a number of papers on the subject, published in the *American Journal of Science.*

62. According to (§ 29) the gyrations of the inner part of a cyclone must be from right to left in the northern hemisphere, and the contrary in the southern, which is the observed law of storms in all parts of the world, as shown by Redfield, and also by Reid, in his *Law of Storms.* It is also evident that at the equator, where $\cos\theta$ vanishes, there cannot be a cyclone, and hence, of all those which Redfield has investigated, and given in his charts of their routes, none have been traced within 10° of the equator. The typhoons or cyclones, also, of the China sea, have never been observed within 9° of the equator.

63. That the atmosphere must run into a gyration, if it converge towards a centre, is evident from the principle demonstrated in (§ 32), by which, in flowing in from all sides towards the centre,

the atmosphere must be deflected to the right in the northern hemisphere, and consequently receive a gyratory motion around that centre, from right to left, and the contrary in the southern hemisphere. Near the equator, this deflecting force vanishes, and consequently there are no cyclones there, as has been shown.

64. Since the atmosphere is depressed in the middle of cyclones, they must sensibly affect the barometer; and this is the true cause of all the great barometrical oscillations, as was first suggested by Redfield. As the cyclone approaches, there is generally a very slight rise of the barometrical column, which is at its maximum at the greatest accumulation near the external part of the cyclone, after which it is gradually depressed, until the middle of the cyclone arrives, where the atmosphere is most depressed, when the barometer is at its minimum, and then it returns in a reverse manner to its former height, when the cyclone has passed. In great storms the mercury sometimes falls more than two inches. In oblong storms, and all imperfectly developed cyclones, the same phenomena must take place in some measure, as in a complete cyclone. We have reason to conclude, therefore, that nearly all the oscillations of the barometer are caused by a cyclonic motion of the atmosphere, by which it is depressed in the middle of the cyclones. The cyclones may be very irregular and imperfectly developed, and not of sufficient violence to produce a strong wind, and several may frequently interfere with one another, so that the oscillations may frequently be very slight ones only, and very irregular.

Since the gyratory motion of a cyclone, and the consequent depression at the centre, depend upon a term containing as a factor, $\cos\theta$, (§ 29), which is the sine of the latitude, according to the preceding theory of barometrical oscillation, the oscillations should be small near the equator, and increase towards the poles, somewhat as the sine of the latitude. Accordingly, at the equator, the mean

monthly range of oscillation is only two millimetres, or less than $\frac{1}{10}$ of an inch, while there is a gradual increase with the latitude; so that at Paris it is 23.66^{mm}, and at Iceland, 35.91^{mm}. (KAEMTZ's *Meteorology, by* C. WALKER, page 297.)

65. The greater rarefaction of the atmosphere at some times than at others, without doubt, has considerable effect upon the barometer; but the theory which attributes the whole of the barometrical oscillations to the rarefaction of the atmosphere produced by the condensation of vapor in the formation of clouds and rain, cannot be maintained; for according to that theory, in the rainy belt near the equator, where there are always copious rains during the day, which are succeeded by a clear atmosphere during the night, the oscillations of the barometer should be greatest, and towards the poles, where there is little condensation of vapor into rain, they should be the least; but we have seen that just the reverse of this is true.

66. When the disturbance of equilibrium is great, but extends over a small area only, the centripetal force is much greater than in the case of large cyclones, and the gyrations are then very rapid and very near the centre, as in the case of tornadoes. Tornadoes generally occur when the surface of the earth is very warm, and the atmosphere calm. For then the strata near the surface become very much rarefied, and are consequently in a kind of unstable equilibrium for a while, when from some slight cause, the rarefied atmosphere rushes up at some point through the strata above, and consequently flows in rapidly from all sides below, and then, unless the sum of all the initial moments of gyration around the centre is exactly equal 0, which can rarely ever be the case, it must run into rapid gyrations near the centre, and a tornado is the consequence. This may be exemplified by the flowing of water through a hole in the bottom of a vessel. If the fluid at the beginning is entirely at rest, it runs out without any gyrations; but if there is the least per-

ceptible initial gyratory motion, it runs into very rapid gyrations near the centre.

67. In the case of tornadoes, which are always of small extent, the influence of the earth's rotation in producing gyratory motions is generally very small in comparison with that of the initial state of the atmosphere, as may be seen by examining equation (42). For if the atmosphere have a very small initial gyratory motion, the term u' depending upon the initial state, will be large in comparison with $n \cos \varphi$ depending upon the earth's rotation, and hence the value of $D_t \mu$, the angular velocity of gyration, depends principally upon the initial gyratory state of the atmosphere with regard to the centre of the tornado, and may be either from right to left, or the contrary. Hence there may be tornadoes at the equator, although here cannot be large cyclones. In large cyclones, the effect of the initial state, except at the equator, is insignificant in comparison with the influence of the earth's rotation; and the latter, moreover, is a constant influence, while the former is soon destroyed by resistances. Hence large cyclones are of long duration, while small tornadoes, depending principally upon the initial gyratory state for their violence, are soon overcome by the resistances.

68. On account of the centrifugal force arising from the rapid gyrations near the centre of a tornado, it must frequently be nearly a vacuum. Hence, when a tornado passes over a building, the external pressure, in a great measure, is suddenly removed, when the atmosphere within, not being able to escape at once, exerts a pressure upon the interior of perhaps nearly fifteen pounds to the square inch, which causes the parts to be thrown in every direction to a great distance. For the same reason, also, the corks fly from empty bottles, and every thing with air confined within, explodes.

69. When a tornado happens at sea, it generally produces a

water-spout. This is generally first formed above, in the form of a cloud, shaped like a funnel or inverted cone. As there is less resistance to the motions in the upper strata than near the earth's surface, the rapid gyratory motion commences there first, when the upper strata of the agitated portion of atmosphere have a tendency to assume somewhat the form of the strata in the case of no resistance, as represented in Fig. 3. This draws down the strata of cold air above, which, coming in contact with the warm and moist atmosphere ascending in the middle of the tornado, condenses the vapor and forms the funnel-shaped cloud. As the gyratory motion becomes more violent, it gradually overcomes the resistances nearer the surface of the sea, and the vertex of the funnel-shaped cloud gradually descends lower, and the imperfect vacuum of the centre of the tornado reaches the sea, up which the water has a tendency to ascend to a certain height, and thence the rapidly ascending spiral motion of the atmosphere carries the spray upward, until it joins the cloud above, when the water-spout is complete. The upper part of a water-spout is frequently formed in tornadoes on land.

When tornadoes happen on sandy plains, instead of water-spouts they produce the moving pillars of sand which are often seen on sandy deserts.

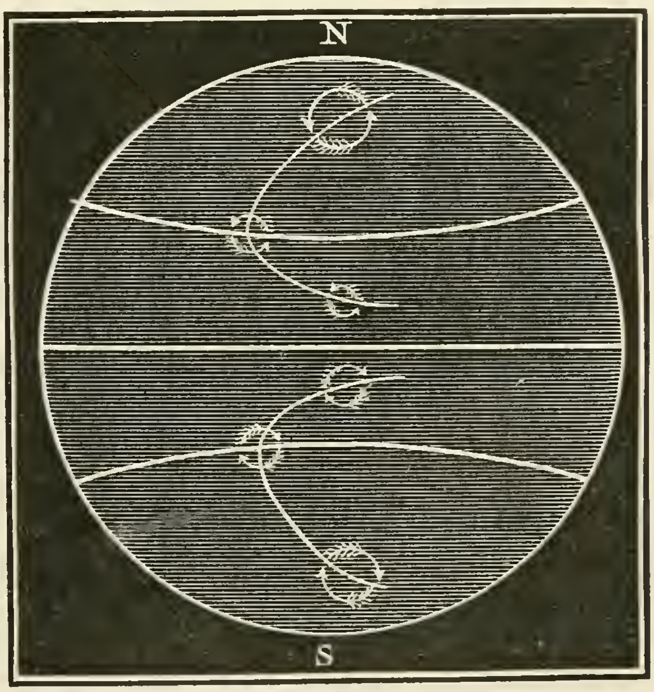

Fig. 7

70. The routes of cyclones in all parts of the world, which have been traced throughout their whole extent, have been found to be somewhat of the form of a parabola, as represented in Fig. 7. Commencing generally near the equator, the cyclone at first

moves in a direction only a little north or south of west, according to the hemisphere, when its route is gradually recurvated towards the east, having its vertex in the latitude of the tropical calm belt, as represented in the figure. This motion of a cyclone may be accounted for by means of what has been demonstrated in (§ 31), which is, that if any body, whether fluid or solid, gyrates from right to left, it has a tendency to move toward the north, but if from left to right, towards the south. Hence the interior and most violent portion of a cyclone, always gyrating from right to left in the northern hemisphere, and the contrary in the southern, must always gradually move towards the pole of the hemisphere in which it is. While between the equator and the tropical calm belt, it is carried westward by the general westward motion of the atmosphere there, but after passing the tropical calm belt, the general motion of the atmosphere carries it eastward, and hence the parabolic form of its route is the resultant of the general motions of the atmosphere, and of its gradual motion toward the pole.

It may be seen from equation (52), that the tendency of a gyrating mass to move towards the pole is as $\sin \psi$, or the cosine of the latitude, and the square of the diameter of the gyrating mass. Hence, near the equator, where the dimensions of the cyclone are always small, it moves slowly toward the pole, but as it gradually increases its dimensions, after passing its vertex, its motion towards the pole, and also its eastward motion, are both increased, and hence its progressive motion in its route or orbit is then accelerated, in accordance with the observations of Redfield.

71. By comparing equations (27) and (44), it is seen that they are very similar, and consequently the motions which satisfy them must be also similar. Hence the general motions of the atmosphere are similar to those of a cyclone. For the general motions of the atmosphere in each hemisphere, form a grand cyclone having the

pole for its centre, and the equatorial calm belt for its limit. But the denser portion of the atmosphere in this case being in the middle instead of the more rare, instead of ascending it descends at the pole or centre of the cyclone.

The southern cyclone having the more rapid motions on account of the resistances from the earth's surface being less, causes a greater depression of the atmosphere there than in the northern cyclone, and throws the calm belt a little north of the equator, as has been explained.

The tendency of the smaller local cyclones, as has been seen, is to run into the centres of the grand hemispherical cyclones, and thus to be swallowed up and become a part of them.

[To be continued.]

THE ELEMENTS OF QUATERNIONS.

By W. P. G. Bartlett, Cambridge, Mass.

[Continued from Page 31.]

III. — Tensors and Versors.

10. If α and β have the same direction, the quaternion $q = \beta \div \alpha$ degenerates into a real and positive number, expressing the numerical ratio of the length of β to that of α, and is then called a *Tensor*. If, in this case, $\alpha = 1$, the tensor $q = \beta \div 1$ expresses the length of the line β, and is called the *tensor of the line* β. The tensor of β is written $T\beta$. The algebraic sum of two or more tensors is evidently a tensor; and, by § 7, a tensor may be applied to any line in space without regard to its direction. Tensors, then, satisfy the condition of § 9, and are commutative in combination with any quaternion.

11. If $T\alpha = T\beta$, the quaternion $q = \beta \div \alpha$ degenerates into the single operation of turning the line α around some axis till it

coincides with β, and is then called a *Versor.* An axis perpendicular to both α and β, and such that rotation around it from the positive direction of α to that of β is positive,* is called the *Axis* of the versor. The angle $\frac{\beta}{\alpha}$ measured positively is called the *Angle* of the versor; and is equal, as usual, to the angle $\left(-\frac{\alpha}{\beta}\right)$.

12. The lines α and β being given, let γ be a line having the same direction as α, and let $\mathrm{T}\gamma = \mathrm{T}\beta$. These conditions completely determine γ. Then, by § 9, as α, β, and γ are co-planar,

$$q = \beta \div \alpha = (\beta \div \gamma)(\gamma \div \alpha) = (\gamma \div \alpha)(\beta \div \gamma);$$

but $\gamma \div \alpha$ is a tensor, and $\beta \div \gamma$ is a versor. Any given quaternion, then, may be resolved into a product of two other determinate quaternions, one of which is a tensor and the other a versor; in this case the former is called the *tensor of the given quaternion,* and the latter *its versor.* The tensor of a quaternion, q, is written $\mathrm{T}q$, and its versor, $\mathrm{U}q$; thus

$$(1) \qquad q = \mathrm{T}q \,.\, \mathrm{U}q = \mathrm{U}q \,.\, \mathrm{T}q.$$

The axis and the angle of $\mathrm{U}q$ may be written $\mathrm{Ax.\,U}q$ and $\angle\,\mathrm{U}q$; or simply $\mathrm{Ax.}q$ and $\angle\,q$, and called the axis and the angle of q. If the axes of three or more quaternions are co-planar, it follows that their planes intersect in a common line. If $\beta = \alpha$, then $q = \alpha \div \alpha = 1$, and $\mathrm{T}q = 1$, $\mathrm{U}q = 1$.

13. If two or more quaternions are equal, their tensors must, by § 12, be equal, and also their versors. As tensors are commutative, we have

$$\Pi q = \mathrm{T}\Pi q \,.\, \mathrm{U}\Pi q = \Pi \mathrm{T}q \,.\, \Pi \mathrm{U}q;$$

but evidently

$$(2) \qquad \mathrm{T}\Pi = \Pi\mathrm{T}, \qquad \text{whence also} \qquad \mathrm{U}\Pi = \Pi\mathrm{U}.$$

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* Either direction of rotation may be arbitrarily assumed as the positive one.

We see, however, from § 8, that we do *not* have in general $T\Sigma = \Sigma T$; and since $U\Sigma q$ must depend on the values of the tensors of the quaternions under the sign Σ, while $\Sigma U q$ does not depend on these values, it is also evident that we do *not* have in general $U\Sigma = \Sigma U$.

14. To determine $\text{Ax.}\,q$, that is, to fix the direction of a line in space, requires two independent elements (such as latitude and longitude, altitude and azimuth, &c.). A quaternion, therefore, involves *four* independent elements, — two to fix its axis, and two more for its angle and tensor; and from this fact its name, *quaternion*, is derived. If two quaternions have their four elements equal each to each, these quaternions must be equal.

15. If in the equations of § 9 we make $\gamma = -\beta$, we have $p = -1$, and $pq = -q$; whence

$$(3)\qquad T(-q) = Tq,\quad \text{Ax.}(-q) = \text{Ax.}\,q,\quad \angle(-q) = 180° + \angle q.$$

If now we consider one axis the negative of another, when it has the opposite direction, a positive rotation around $\text{Ax.}\,q$ is equivalent to an equal amount of negative rotation around $-\text{Ax.}\,q$; therefore instead of (3) we may have

$$(3')\quad T(-q) = Tq,\quad \text{Ax.}(-q) = -\text{Ax.}\,q,\quad \angle(-q) = 180° - \angle q.$$

(4) 16. If $\qquad Tp = Tq,\quad \text{Ax.}\,p = -\text{Ax.}\,q,\quad \angle p = \angle q,$

or, what is the same thing, if

$$(4')\qquad Tp = Tq,\qquad \text{Ax.}\,p = \text{Ax.}\,q,\qquad \angle p = -\angle q,$$

then p is called the *Conjugate* of q, and is written $p = Kq$.

If $p = Kq$, evidently

$$(5)\qquad q = Kp = KKq = K^2 q,\qquad q \cdot Kq = Kq \cdot q = (Tq)^2;$$

and by substituting Kq for q in (1) we have, since, by (4), $UK = KU$,

$$(6)\qquad Kq = TKq \,.\, UKq = Tq \,.\, KUq;$$

and by making the same substitution in (3)

$$T(-Kq) = TKq = Tq, \quad \text{Ax.}(-Kq) = \text{Ax.}\,Kq = -\,\text{Ax.}\,q,$$
$$\angle(-Kq) = 180^\circ + \angle Kq = 180^\circ + \angle q;$$

but $TK(-q) = T(-q) = Tq$, $\text{Ax.}K(-q) = -\text{Ax.}(-q) = -\text{Ax.}q$,
$$\angle K(-q) = \angle(-q) = 180^\circ + \angle q;$$

(7) whence, by § 14, $\qquad K(-q) = -\,Kq.$

17. Let $T\alpha = T\beta = T\gamma$, and denote the versors, $\beta \div \alpha$, $\gamma \div \beta$, $\gamma \div \alpha$, respectively by u, u', u''. Then if

$$(\gamma \div \beta)\,(\beta \div \alpha) = \gamma \div \alpha = u'u = u'',$$

also $\qquad (\alpha \div \beta)\,(\beta \div \gamma) = \alpha \div \gamma = Ku\,.\,Ku' = Ku'';$

whence, by (6),

(8) $\;Kpq^{*} = Tpq\,.\,KUpq = Tq\,.\,Tp\,.\,KUq\,.\,KUp = Kq\,.\,Kp.$

As taking the conjugate of a quaternion amounts merely to changing the arbitrarily assumed positive direction of rotation to its opposite, we have

(9) $\qquad K\Sigma = \Sigma K.$

18. If any number of quaternions have a common axis, this axis will evidently be the axis of their product, and the angle of their product will be the algebraic sum of their angles. If all these factors are equal their product will be a *power* of a quaternion. Negative and fractional powers are defined by the equations

(10) $\qquad q^{-m}\,q^{m} = 1 \qquad$ and $\qquad q^{\frac{m}{n}}\,q^{n} = q^{m},$

or $\qquad q^{-m} = 1 \div q^{m} \qquad$ and $\qquad q^{\frac{m}{n}} = q^{m} \div q^{n},$

in which m and n are real integer numbers. Evidently, for all real values of m, integral or fractional,

* The conjugate of pq is here written Kpq. HAMILTON uses $K\,.\,pq$. The product of q multiplied by Kp is uniformly written $Kp\,.\,q$. The same distinction is made with all symbols employed.

(11) $\mathrm{T}\,q^{m}$* $=(\mathrm{T}\,q)^m$, $\mathrm{U}\,q^m=(\mathrm{U}\,q)^m$, $\mathrm{Ax.}\,q^m=\mathrm{Ax.}\,q$, $\angle q^m = m\angle q$.†

After substituting q^m for q in (4), and $\mathrm{K}\,q$ for q in (11), a comparison of these equations leads, by a similar process to that used in obtaining (7), to the equation

(12) $$\mathrm{K}\,q^m = (\mathrm{K}\,q)^m.$$

The first of equations (10) is equivalent, by §§ 12 and 13, to the two equations

(10′) $$\mathrm{T}\,q^{-m}\,.\,\mathrm{T}\,q^m = 1, \qquad \text{and} \qquad \mathrm{U}\,q^{-m}\,.\,\mathrm{U}\,q^m = 1;$$

whence, since, by (4), $\mathrm{U}\,q^m\,.\,\mathrm{U\,K}\,q^m = 1$, it follows that

(13) $$\mathrm{U}\,q^{-m} = \mathrm{U\,K}\,q^m;$$ whence also, by (10′),

(14) $$q^{-m} = \mathrm{T}\,q^{-m}\,.\,\mathrm{U}\,q^{-m} = \mathrm{T}\,q^{-m}\,.\,\mathrm{U\,K}\,q^m = \mathrm{T}\,q^{-2m}\,.\,\mathrm{K}\,q^m.$$

By putting $p\,q$ for q and $m=1$ in (14), we get by (8) and (14),

(15) $$(p\,q)^{-1} = (\mathrm{T}\,p\,q)^{-2}\,[\mathrm{K}\,p\,q = \mathrm{T}\,q^2\,.\,q^{-1}\,.\,\mathrm{T}\,p^2\,.\,p^{-1}] = q^{-1}\,.\,p^{-1}.‡$$

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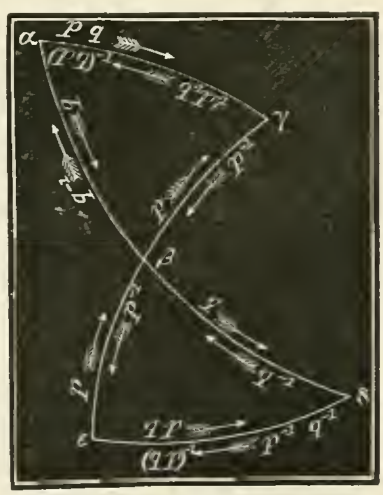
* The tensor of q^m is here written $\mathrm{T}\,q^m$, and the m^{th} power of $\mathrm{T}\,q$, $(\mathrm{T}\,q)^m$. HAMILTON uses $\mathrm{T}\,.\,q^m$ and $\mathrm{T}\,q^m$ respectively. The same distinction is made with all symbols employed.

† It should be observed, that, when m is fractional, since $\angle q = t\,360° + \angle q$, $\angle q^m = m\,(\angle q + t\,360°)$; and in such cases $\angle q^m$ will have a certain number of different values for different integral values of t. For convenience, however, we suppose $t=0$, when nothing is said to the contrary, so that $\angle q^m = m\angle q$ in all cases.

‡ The geometry of this and some previous propositions may be shown as follows, *supposing p and q to be only versors.* Let a sphere be described with a radius of unity from the common origin of the lines α, &c. as a centre. Let the intersections of this sphere with these lines be connected by great circle arcs. Then, by §§ 5, 7, and 9, we may represent q by either of the equal arcs, $\alpha\beta$, or $\beta\delta$, and p by $\varepsilon\beta$, or $\beta\gamma$. Then $p\,q$ is represented by $\alpha\gamma$, and $q\,p$ by $\varepsilon\delta$. Equations (4′) and (11) show that $\mathrm{K}\,q$ and q^{-1} are equal, *when q is a versor*, and that either may be represented by $\beta\alpha$ or $\delta\beta$. A simple examination of the accompanying figure shows the meaning of (15) and §§ 7 and 9, *as far as versors are concerned;* and as tensors are mere numerical elements following the ordinary rules of arithmetic, they add no difficulty to these cases.



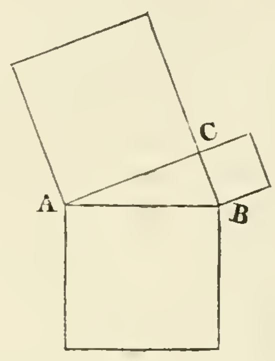
A SECOND BOOK IN GEOMETRY.

[Continued from Page 410, Vol. I.]

CHAPTER VI.

THE PYTHAGOREAN PROPOSITION.

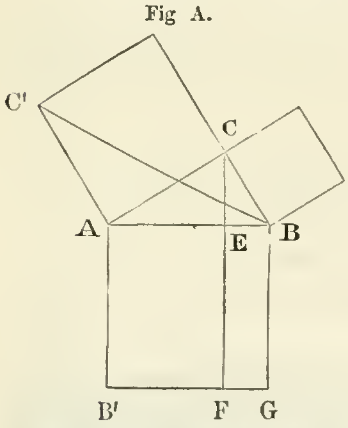
64. We recollect that the square built on the hypotenuse of a right triangle is equivalent in its area to the sum of the squares built upon its legs. This is one of the most useful of all geometrical truths. Let us first analyze it in one or two modes, and then build it up synthetically by the same paths. We may afterwards, if we like, devise other modes of analysis and synthesis, for this proposition, like all others, may be approached in various ways.



65. The Pythagorean proposition or theorem might be suggested in different ways. But in whatever way we were led to suspect that the square on the hypotenuse is equivalent to the sum of the squares on the legs, we should, in reflecting upon it, probably begin by drawing a right triangle with a square built upon each side.

66. We should inquire whether the square on the hypotenuse could be divided into two parts that should be respectively equal to the other two squares. And we should judge that these parts should be somewhat similar to each other in shape, because the legs do not differ in their relations to the hypotenuse except in size, and in the angles they make with it.

Fig A.



67. But we cannot readily conceive of any division of the square into two somewhat similar parts, except into two rectangles. And then it is apparent that two rectangles bearing respectively the same relations to the squares on the legs, may be found by drawing a line from the vertex of the right angle at right angles with the hypotenuse, and continuing it through the square, as C F is here drawn.

68. It will now only be necessary to show that one of these rectangles is equivalent to its corresponding square; because the same mode of proof will obviously answer for the other rectangle and its square.

69. Now if we know, or can prove, that the area of a rectangle is measured by the product of its sides, we shall have to prove that $AE \times AB'$ or $AE \times AB$ is equivalent to $AC \times AC'$.

70. But by the doctrine of proportion it may be shown that this would be equivalent to saying that A E is to A C as A C is to A B.

71. Again, it may be shown by geometry that this proportion between the lines A B, A C, and A E, would be true if the triangle A E C were similar to A C B, and that A E stood in one to A C, as A C stood to A B in the other; so that all that remains for us to do is to show that these triangles are similar.

72. But we can show by geometry that two triangles are similar when their angles are equal.

73. And it is easy to show that the angles of these triangles are equal to each other.

74. For C A B and C A E are the same angle; A C B and A E C are both right angles, and therefore A B C and A C E are each complements of C A E. Moreover, A C and A E are situated in the triangle A E C, in the same manner that A B and A C are situated in the triangle A B C.

75. We have thus, in articles 66 – 74, sufficiently analyzed the Pythagorean proposition to enable us to build it up again in a deductive form. This analysis, however, has been partly algebraical, as it has introduced the idea of multiplying two lines to produce a surface. Let us now begin and build up the proposition by the same road. We shall find 31 articles necessary, and I will number them from 76 to 106.

First Proof of the Pythagorean Proposition.

76. *Definition.* The comparative size of two quantities is called their ratio; thus if one is twice as large as the other, they are said to be in the same ratio as that of 2 to 1; or to be in the ratio 2 to 1; or it is said, in a looser way, that their ratio equals 2.

77. *Notation.* Ratio is written by means of the marks :, $\div$, and by writing one quantity over the other. Thus, A : B, A $\div$ B, and $\frac{A}{B}$, are each used to signify the ratio of A to B. These marks are the same as those used in arithmetic to signify Quotient, because the meaning of a quotient is " a number having the same ratio to one, that the dividend has to the divisor." The ratio of A to B is not the quotient of A divided by B, but it is the ratio of that quotient to unity.

78. *Axiom.* If each of two quantities is multiplied or divided by the same number, the ratio of the products or quotients will be the same as that of the quantities themselves. Thus twenty inches is in the same ratio to twenty rods as one inch to one rod, or as the twentieth of an inch to the twentieth of a rod.

79. *Definition.* A proportion is the equality of two ratios. Thus (if we use the sign = to signify " is equal to") A : B = C : D is the statement of a proportion. It signifies that A is in the same proportion to B that C is to D.

80. *Definition.* When a proportion is written as in article 79, the first and last terms, that is, A and D, are called the extremes, and the others, that is, B and C, are called the means.

81. *Theorem.* In every proportion the product of the means is equal to that of the extremes. — *Proof.* In any proportion, as M : N = P : Q, we wish to prove (using the mark $\times$ to signify "multiplied by") that $M \times Q = N \times P$. Now in order to do this, we must use only self-evident truths. The only truth of this character that we have given above is that of article 78. But in order, by means of the multiplications of article 78, to change the first ratio M : N into $M \times Q$, we must, whatever else we do, at least multiply each term by Q, and this will give us $M \times Q : N \times Q = P : Q$, and in order to change the second ratio P : Q into $N \times P$, we must, at all events, multiply each term by N, and this will give us $M \times Q : N \times Q = N \times P : N \times Q$.

Thus from the self-evident truth of article 78 we find that the product of the means bears the same ratio to the product $N \times Q$ that is borne to it by the product of the extremes. And as it is self-evident that two quantities, bearing the same ratio to a third, must be equal to each other, we have proved that the product of the means is equal to that of the extremes.

82. *Definition.* When both the means are the same quantity, that quantity is called a mean proportional between the extremes.

83. *Corollary.* It follows from article 81, that the product of the mean proportional multiplied by itself is equal to the product of the extremes.

84. *Definitions.* A *unit* of length is a line taken as a standard of comparison for lengths. Thus an inch, a foot, a pace, a span, etc., are units. The *length* of any line is its ratio to the unit of length.

85. *Definition.* A unit of surface is a surface taken as a standard of comparison. The most common unit of surface is a square whose side is a unit of length.

86. *Definition.* The area of a surface is the ratio of the surface to the unit of surface.

87. *Theorem.* Any straight line in the same plane with two parallel lines makes the same angle with one that it does with the other. — *Proof.* For as the straight line has but one direction, and each of the parallel lines may always be considered as going in the same direction as the other, the difference of that direction from the direction of the third straight line must be the same for each of the parallel lines.

88. *Corollary.* If a straight line is parallel to one of two parallel lines, it is parallel to the other; if at right angles to one of the two, it is at right angles to the other.

89. *Theorem.* If a straight line make on the same side of itself the same angle with two other straight lines in the same plane, those other straight lines must be parallel.

Scholium. The line must not be conceived as reversing its direction at any point. — *Proof.* For if two directions differ equally from a third, they must be equal to each other.

90. *Axiom.* If the boundaries of one plane surface are similar to those of another in such a way that the two surfaces would coincide in extent if laid one upon the other, the two surfaces are equivalent.

[To be continued.]

Editorial Items.

R. T. Massie, Esq., late Assistant Instructor in Mathematics in the University of Virginia, has accepted the Professorship of Mathematics in Randolph-Macon College, Va.

Daniel Vaughan, Esq., has accepted the Professorship of Mathematics in Masonic College, at Lagrange, Ky.

Prof. William H. Young has been transferred from the Professorship of Mathematics to that of Ancient Languages in the Ohio University at Athens. Eli T. Tappan, of Steubenville, Ohio, has been appointed to the Professorship of Mathematics thus vacated.

Prof. J. W. Patterson, of Dartmouth College, has been transferred from the Professorship of Mathematics to that of Astronomy, made vacant by the decease of Prof. Ira Young.

All solutions and communications for the Monthly should be addressed to the Editor, at Cambridge, Mass., and all remittances and business letters should be sent to Messrs. Ivison and Phinney, 48 and 50 Walker St., New York. A compliance with this request will save us much trouble.

Erratum. — In Dem. 19, p. 49, for B T U K read B T U H.

Books Received. — Place of Mathematics in University Education. Inaugural Address of Charlton S. Lewis, Professor of Pure Mathematics in Troy University, delivered before the Trustees at their annual meeting, July 20th, 1859. — Nouvelles Annales de Mathématiques. Octobre, 1859. — A Descriptive Catalogue of Text-Books for Schools and Colleges. Published by Ivison and Phinney, 48 and 50 Walker St., New York. Edition of September, 1859, 8vo. pp. 168. Gratis to teachers and those especially interested in Education. This volume is filled with recommendations of the Publishers' series of books. They express the deliberate opinions of educated men in all parts of the country. Teachers will do well to procure a copy of this Catalogue for reference; as it contains the idea of a large number of practical teachers as to what constitutes a good Text-Book.

THE

MATHEMATICAL MONTHLY.

Vol. II. . . . JANUARY, 1860. . . . No. IV.

PRIZE PROBLEMS FOR STUDENTS.

I. A and B can do a piece of work in m days; B and C in n days; in what time can A and C do the same, it being supposed that A can do p times as much as B in a given time?

II. Show that $\frac{2+\sqrt{3}}{\sqrt{2}+\sqrt{(2+\sqrt{3})}}+\frac{2-\sqrt{3}}{\sqrt{2}-\sqrt{(2-\sqrt{3})}}=\sqrt{2}$.

III. Find the roots of the equation $x^3 - 6x = 4$, by trigonometry.

IV. Four persons, A, B, C, D, in order, beginning with A, cut a pack of cards, replacing them after each cut, on condition that the first who cuts a heart shall win. What are their respective probabilities of success?

V. The notation of Problem V. in the November number of the Monthly being retained, prove that in the plane

$$\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C = \cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C;$$

and in the sphere

$$\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C = \cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C\,\frac{\cos(\rho+\delta)\cos(\rho-\delta)}{\cos^2 r\cos^2\rho},$$

$$\cot\tfrac{1}{2}A\,\frac{\cos\delta'}{\cos\rho'} + \cot\tfrac{1}{2}B\,\frac{\cos\delta''}{\cos\rho''} + \cot\tfrac{1}{2}C\,\frac{\cos\delta'''}{\cos\rho'''} = \cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C\,\frac{\cos\delta}{\cos\rho}.$$

The solutions of these problems must be received by March 1, 1860.

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. I., Vol. II.

The first Prize is awarded to J. D. Van Buren, Rensselaer Polytechnic Institute, Troy, N. Y.

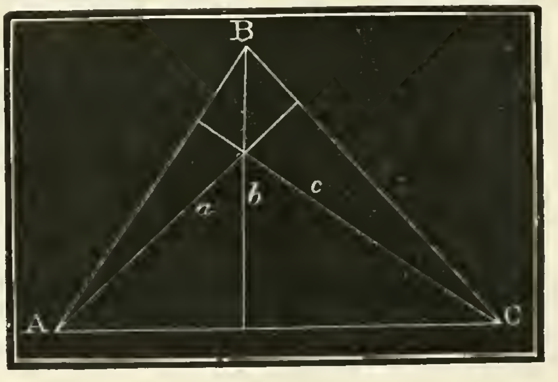
The second Prize is awarded to O. B. Wheeler, Sophomore class, University of Michigan, at Ann Arbor.

The third Prize is awarded to George H. Tower, Classical High School, Petersham, Mass.

Prize Solution of Problem II.

By Perrin B. Page, Nunda, N. Y.

In any plane triangle, prove that the sines of the angles are inversely as the perpendiculars let fall from them upon the opposite sides.



Let $A\,B\,C$ be the triangle, and denote by a, b, c, the perpendiculars dropped respectively from the angles A, B, C. By trigonometry

(1.) $$\frac{\sin A}{B\,C} = \frac{\sin B}{A\,C} = \frac{\sin C}{A\,B}.$$

By geometry, twice the area of $A\,B\,C$ is

(2.) $$B\,C \times a = A\,C \times b = A\,B \times c.$$

Multiplying equations (1) and (2) together, member by member, gives,

$$\sin A \times a = \sin B \times b = \sin C \times c\text{; or,}$$

$$\sin A : \sin B : \sin C = c : b : a.$$

Second Solution. — By definition,

$$\sin A = \frac{b}{A\,B} = \frac{c}{A\,C}, \qquad \sin B = \frac{a}{A\,B} = \frac{c}{B\,C},$$

$$\therefore \frac{\sin A}{\sin B} = \frac{b}{a} = \frac{B\,C}{A\,C}.$$

$$\therefore \sin A : \sin B = b : a = BC : AC,$$

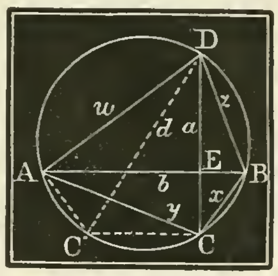
which not only proves the proposition, but also that *the sines of the angles are proportional to their opposite sides.* This is a slight modification of W. C. Henck's solution.

Prize Solution of Problem III.

By Isaac H. Turrell, Mt. Carmel, Indiana.

Having given the diagonals of a quadrilateral inscribed in a given circle to determine its sides geometrically, when the diagonals intersect each other at right angles. — Communicated by Professor D. W. Hoyt.

Let AB and CD be the given diagonals. Inscribe AB in the given circle. Next, draw parallel to AB and at a distance from the centre equal to one half the other diagonal CD, the line CC'. From either extremity of this line, as C, draw a line perpendicular to AB and produce it till it meets the circumference in D. The line CD is therefore inscribed in the circle, perpendicular to AB, and $ACBD$ is the required quadrilateral. Charles B. Boutelle's solution is essentially the same as the above.



Prize Solution of Problem IV.

By J. D. Van Buren, Rensselaer Polytechnic Inst., Troy, N. Y.

Given

$$(1.)\quad xw + yz = ab, \qquad (2.)\quad xy + zw = ad$$

$$(3.)\quad xz + yw = bd, \qquad (4.)\quad x^2 + w^2 = y^2 + z^2;$$

to find the values of x, y, z, and w. — Communicated by Professor D. W. Hoyt.

Adding and subtracting (1) and (3), (1) and (2), and also multiplying (3) and (2) together, we obtain

$$(5.)\quad x + y = \frac{bd + ab}{z + w}, \qquad (7.)\quad x + z = \frac{ab + ad}{w + y},$$

$$(6.)\quad x - y = \frac{bd - ab}{z - w}, \qquad (8.)\quad x - z = \frac{ab - ad}{w - y},$$

$$(9.) \qquad y z (x^2 + w^2) + x w (z^2 + y^2) = a b d^2.$$

Combining the products of (5) and (6), and (7) and (8), with (9) and (4) respectively, we get

$$x^2 - y^2 = \pm b \sqrt{d^2 - a^2}, y^2 - w^2 = \pm a \sqrt{b^2 - d^2}, x^2 + w^2 = d^2 = y^2 + z^2.$$

Therefore,

$$2 x^2 = d^2 \pm b \sqrt{d^2 - a^2} \pm a \sqrt{b^2 - d^2},$$

$$2 y^2 = d^2 \mp b \sqrt{d^2 - a^2} \pm a \sqrt{b^2 - d^2},$$

$$2 w^2 = d^2 \mp b \sqrt{d^2 - a^2} \mp a \sqrt{b^2 - d^2},$$

$$2 z^2 = d^2 \pm b \sqrt{d^2 - a^2} \mp a \sqrt{b^2 - a^2}.$$

Whence we obtain the values of the required quantities by dividing by 2 and extracting the square root. Similar solutions were given by William W. Johnson and O. B. Wheeler.

O. B. Wheeler remarks that the equations of Problem IV. are readily obtained from the figure of Problem III. From geometry we have, "*The rectangle of the two diagonals is equivalent to the sum of the rectangles of the opposite sides,*" which is (1). We have (2) and (3) by the proposition, "*In any triangle the rectangle contained by two sides is equivalent to the rectangle contained by the diameter of the circumscribed circle and the perpendicular let fall on the third side;*" and (4) from the sum of the equations

$$A E^2 = w^2 - D E^2 = y^2 - E C^2,$$

$$B E^2 = x^2 - E C^2 = z^2 - D E^2.$$

The connection between Problems III. and IV. was also noticed by W. C. Cleveland, who gave the following geometrical demonstration of (4).

Let the chord $C C'$ be drawn parallel to $A B$; then $A C' = B C = x$, angle $C' C D = 90°$;

$\therefore$ $C' D$ is a diameter; $\qquad \therefore$ $x^2 + w^2 = d^2$.

Similarly $y^2 + z^2 = d^2$; whence equation (4).

Prize Solution of Problem V.

By J. D. Van Buren, Rensselaer Polytechnic Institute, Troy, N. Y.

If, in a plane or spherical triangle, A, B, C denote the angles, and a, b, c the opposite sides respectively; if r, ρ denote the radii of the circumscribed and inscribed circles, and δ the distance between the centres of these circles; then in the plane triangle

$$\sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B \cos \tfrac{1}{2} C,$$

and in the spherical triangle

$$\sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B \cos \tfrac{1}{2} C \frac{\cos \delta}{\cos r \cos \rho}.$$

Plane Triangle. We have from plane trigonometry,

$$\sin A = \frac{2k}{bc}, \quad \sin B = \frac{2k}{ac}, \quad \sin C = \frac{2k}{ab},$$

when $k = \sqrt{s(s-a)(s-b)(s-c)}$, and $s = \frac{1}{2}(a+b+c)$.

$$\therefore \sin A + \sin B + \sin C = \frac{2k(a+b+c)}{abc} = \frac{4ks}{abc}.$$

Also, $\cos \frac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}}$, $\cos \frac{1}{2} B = \sqrt{\frac{s(s-b)}{ac}}$, $\cos \frac{1}{2} C = \sqrt{\frac{s(s-c)}{ab}}$.

$$\therefore \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B \cos \tfrac{1}{2} C = \frac{ks}{abc},$$

$$\therefore \sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B \cos \tfrac{1}{2} C.$$

Spherical Triangle. We have, from spherical trigonometry,

$$\sin A = \frac{2k}{\sin b \sin c}, \quad \sin B = \frac{2k}{\sin a \sin c}, \quad \sin C = \frac{2k}{\sin a \sin b},$$

in which

$k = \sqrt{\sin s \sin(s-a) \sin(s-b) \sin(s-c)}$, and $s = \frac{1}{2}(a+b+c)$.

(1.) $$\therefore \sin A + \sin B + \sin C = \frac{2k(\sin a + \sin b + \sin c)}{\sin a \sin b \sin c}.$$

Also,

$$\cos \tfrac{1}{2} A = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}, \quad \cos \tfrac{1}{2} B = \sqrt{\frac{\sin s \sin(s-b)}{\sin a \sin c}},$$

$$\cos \tfrac{1}{2} C = \sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}}.$$

$$\therefore \cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C = \frac{k\sin s}{\sin a\sin b\sin c}.$$

But we have (CHAUVENET'S Trig., p. 251, and Eq. 319),

$$\frac{\cos\delta}{\cos r\sin\varrho} = \frac{\sin a + \sin b + \sin c}{2k}, \qquad \tan\varrho = \frac{k}{\sin s},$$

$$\therefore \sin\varrho = \frac{k\cos\varrho}{\sin s} \qquad \therefore \frac{\cos\delta}{\cos r\cos\varrho} = \frac{\sin a + \sin b + \sin c}{2\sin s}.$$

$$(2.) \quad \therefore \cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C\,.\,\frac{\cos\delta}{\cos r\cos\varrho} = \frac{k(\sin a + \sin b + \sin c)}{2\sin a\sin b\sin c}.$$

Therefore, by comparing (1) and (2) we get

$$\sin A + \sin B + \sin C = 4\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C\,.\,\frac{\cos\delta}{\cos r\cos\varrho}.$$

SECOND SOLUTION.

By O. B. WHEELER, University of Michigan.

Plane Triangle. Since in a plane triangle $\tfrac{1}{2}(A + B + C) = \tfrac{1}{2}\pi$, we have

(1.) $\cos\tfrac{1}{2}(A + B + C) = \cos\tfrac{1}{2}\pi = 0.$

(2.) $\cos\tfrac{1}{2}(-A + B + C) = \cos(\tfrac{1}{2}\pi - A) = \sin A,$

(3.) $\cos\tfrac{1}{2}(A - B + C) = \cos(\tfrac{1}{2}\pi - B) = \sin B,$

(4.) $\cos\tfrac{1}{2}(A + B - C) = \cos(\tfrac{1}{2}\pi - C) = \sin C.$

Taking the sum of (1) and (2), and also of (3) and (4), we obtain

(5.) $2\cos\tfrac{1}{2}A\cos\tfrac{1}{2}(B + C) = \sin A,$

(6.) $2\cos\tfrac{1}{2}A\cos\tfrac{1}{2}(B - C) = \sin B + \sin C.$

Adding (5) and (6), developing and reducing, we have

$$4\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C = \sin A + \sin B + \sin C.$$

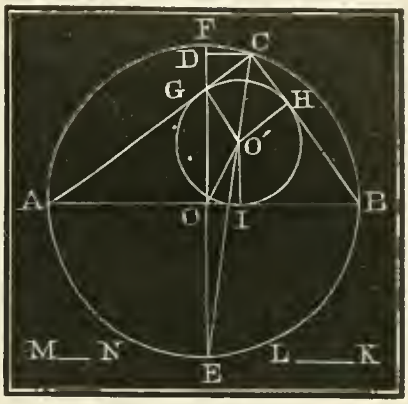
MR. WHEELER'S solution for the spherical triangle is essentially the same as the one already given.

SIMON NEWCOMB.
W. P. G. BARTLETT.
TRUMAN HENRY SAFFORD.

ANOTHER SOLUTION OF PRIZE PROBLEM I., No. IX.

By GEORGE EASTWOOD, Saxonville, Mass.

THERE are two ways of solving a geometrical problem *geometrically*. One way is to solve it by *synthesis*, the other way is to solve it by *analysis*. In solving a problem by analysis, we assume that the work *is done;* and then proceed, as the word implies, to take it to pieces, and to examine and develop the properties of each part and its relations to all the other parts. To solve the same problem by synthesis, we proceed to construct it upon elementary principles, from the given data and such combinations of them as the exigencies of the case may require. We have a beautiful illustration of the synthetic method in Mr. EVANS's solution of Prize Problem I. of No. IX., published in the last Monthly. The following solution of the same problem is offered to young students as a simple specimen of the analytic method.*



Analysis. Suppose the thing done, that ACB is the required triangle, and that $AFBE$, GHI, are its circumscribing and inscribed circles. Join O, O', the centres of the circles, and draw the radii $O'G$, $O'H$, $O'I$, to the points of contact G, H, I. Draw the diameter EF perpendicular to the hypothenuse AB, and CD perpendicular to EF.

By the question, we have given,

$$AC - BC = (AG + GC) - (BH + HC)$$
$$= AI - BI = 2\,OI = 2\,MN \text{ suppose.}$$

* Strictly speaking, the two ways of solving a geometrical problem, above indicated, are not two distinct and independent methods of doing the same thing, but rather the different parts of one full and perfect method. The one process is the reverse of the other, and the use of each is indispensable to a complete solution.

Also, we have given,

$$AB - BC = (AI + IB) - (IB + IO') = AI - IO',$$

which suppose equal to $2\,LK$. Take $2\,MN$ from $4\,LK$, then

$$AI + BI - 2\,IO' = 4\,LK - 2\,MN;$$
$$\therefore\ AO = 2\,LK - MN + IO'.$$

By Prop. VI., *Liverpool Student*, $EO \,.\, FD = OI^2 = MN^2$; so that when $EO = AO$ is found, FD will be given.

By Prop. 47, Euc. I.

$$OI^2 + O'I^2 = O'O^2 = AO\,(AO - 2\,O'I)$$
$$= (2\,LK - MN)^2 - O'I^2 = (2\,LK - OI)^2 - O'I^2,$$
$$\therefore\ O'I^2 = 2\,LK(LK - MN).$$

As LK and MN are both given lines, therefore $O'I$ is given. Hence this

Construction. Find (Euc. VI. 13) $O'I$ a mean proportional between $2\,LK$ and $LK - MN$, and with centre O and radius $OA = 2\,LK - MN + O'I$, describe a circle. Draw the horizontal and vertical diameters AB, FE, and upon FE apply (Euc. VI. 11) FD a third proportional to EO, OI. Draw DC parallel to AB to meet the circle in C, and join AC, BC; then ACB is the required triangle. The demonstration is evident from the analysis.

NOTES AND QUERIES.

1. *Note on Decimals.*—Decimals should be taught in written arithmetic in connection with whole numbers. Let them be treated in the same manner as whole numbers, and not as common fractions, there being no necessity for confusing the mind of the pupil by writing the denominator. In 31.3 there is no more need of indicating that the three standing in tenths' place is divided by ten, than

that the three standing in ten's place is multiplied by ten. The pupil can readily be made to understand these relations without the divisor or multiplier. There should be no separate divisions in arithmetic for Decimal Fractions and Federal Money. The four simple rules should embrace these, and Percentage and Interest should be treated immediately after as a development of the decimal notation. The tables and Compound Numbers should immediately precede Vulgar Fractions,—the former *in*creasing in an irregular ratio, and the latter *de*creasing in an irregular ratio. If authors in writing arithmetics would observe this order, it would result in great advantage to the learner.—Samuel P. Bates, Superintendent of Public Instruction, Crawford County, Pa.

2. *Reduction of Fractions to a Common Denominator.*—After the pupil has thoroughly learned that multiplying or dividing both terms of a fraction by the same number does not change its value, he is then prepared to learn how to reduce fractions to a common denominator. The following process is very simple, and the pupil should be required to repeat it until it is thoroughly understood.

Let it be required, for example, to reduce $\frac{7}{14}$, $\frac{9}{15}$, $\frac{23}{28}$, $\frac{19}{35}$, $\frac{12}{21}$ to a common denominator. First, resolve the terms of all the fractions into their prime factors, and write them in a horizontal row. Now it is evident that when the denominators are common they will contain the same factors.

1st.	2d.	3d.	4th.	5th.
$\frac{7}{2 \cdot 7}$	$\frac{3 \cdot 3}{3 \cdot 5}$	$\frac{23}{2 \cdot 2 \cdot 7}$	$\frac{19}{5 \cdot 7}$	$\frac{2 \cdot 2 \cdot 3}{3 \cdot 7}$
$\frac{7}{2 \cdot 7}$	$\frac{3 \cdot 3 \cdot 7}{3 \cdot 5 \cdot 7}$	$\frac{23}{2 \cdot 2 \cdot 7}$	$\frac{19}{5 \cdot 7}$	$\frac{2 \cdot 2 \cdot 3}{3 \cdot 7}$
$\frac{7}{2 \cdot 7}$	$\frac{3 \cdot 7}{5 \cdot 7}$	$\frac{23}{2 \cdot 2 \cdot 7}$	$\frac{19}{5 \cdot 7}$	$\frac{2 \cdot 2}{7}$
$\frac{7 \cdot 5}{2 \cdot 7 \cdot 5}$	$\frac{3 \cdot 7}{5 \cdot 7}$	$\frac{23 \cdot 5}{2 \cdot 2 \cdot 7 \cdot 5}$	$\frac{19}{5 \cdot 7}$	$\frac{2 \cdot 2 \cdot 5}{7 \cdot 5}$
$\frac{2 \cdot 7 \cdot 5}{2 \cdot 2 \cdot 7 \cdot 5}$	$\frac{2 \cdot 2 \cdot 3 \cdot 7}{2 \cdot 2 \cdot 5 \cdot 7}$	$\frac{23 \cdot 5}{2 \cdot 2 \cdot 7 \cdot 5}$	$\frac{2 \cdot 2 \cdot 19}{2 \cdot 2 \cdot 5 \cdot 7}$	$\frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 5}{2 \cdot 2 \cdot 7 \cdot 5}$

Let us first compare them with reference to the factor 7. This factor is found in all the denominators but one; and in order

to make them common with reference to 7, it must either be introduced into the denominator of the 2d fraction, or removed from the denominators of all the other fractions. But to remove a factor from the denominator of a fraction without changing its value, it must be removed from the numerator also. Now 7 is a factor in only one of the numerators, and the denominators cannot be made common by removing the 7. It must therefore be introduced into the denominator of the second fraction to make the denominators common; and also into the numerator, in order not to change the value of the fraction. We thus get the second row. Next compare the denominators with reference to the factor 3. It is found in only two of them, from which it may be removed, since it is also found in the corresponding numerators. Thus we get the third row. (It is plain, that instead of removing the factor 3 from the 2d and 5th fractions, the denominators might be made common with reference to 3 by introducing it into the terms of all the other fractions. We should not, however, in this way reduce the fractions to their least common denominator.) Next introduce the factor 5 and get the fourth row; and lastly, introduce the factors 2.2, which makes all the denominators common. — Teacher.

3. *Note on the superior limit of the Roots of an Equation.*

Prop. I. *The greatest negative coefficient of an equation, plus unity, is a superior limit of its roots.*

Proof. Let us assume the general equation,

$$x^n \pm Ax^{n-1} \pm Bx^{n-2} \ldots\ldots \pm Nx^{n-r+1} \pm V = 0.$$

A superior limit of the roots of an equation must produce a positive result when substituted for x; that is, the sum of all the positive terms must exceed the sum of all the negative ones by some quantity R. The most unfavorable case evidently is that in which *all the terms after the first are negative and have equal coefficients.* Under this assumption our equation becomes

$$x^n - Nx^{n-1} - Nx^{n-2} \ldots\ldots - N = 0.$$

Suppose a to be greater than the greatest root, then

$$a^n - Na^{n-1} - Na^{n-2} \ldots\ldots - N = R; \text{ or}$$

$$a^n - N(a^{n-1} - a^{n-2} \ldots\ldots - 1) = R; \text{ or}$$

$$a^n - N\left(\frac{a^n - 1}{a - 1}\right) = R.$$

$$\therefore a^n = N\left(\frac{a^n - 1}{a - 1}\right) + R = \frac{Na^n}{a - 1} - \frac{N}{a - 1} + R.$$

Now suppose a to have been so taken that $\frac{N}{a-1} = R$, as may always be done; then we shall have

$$a^n = \frac{Na^n}{a - 1}, \qquad \text{or } a - 1 = N, \qquad \text{or } a = N + 1.$$

But $a > x$, $\therefore N + 1 > x$; and hence $N + 1$ is a superior limit, as was to be proved.

Prop. 2. *If unity be added to that root of the greatest negative coefficient, which is denoted by the number of terms preceding the first negative term, the result will be a superior limit of the roots of the equation.*

In the equation

$$x^n + Ax^{n-1} + Bx^{n-2} + \ldots - Kx^{n-r} \pm \ldots - Px^{n-r-s} \ldots \pm V = 0,$$

suppose K to be the *first*, and P the *greatest*, negative coefficient, and r the number of terms which precede the first negative term. Evidently the most unfavorable case is that in which the coefficients A, B, C, &c., preceding K are zero, and all the coefficients K, L, M, &c., are all negative and equal. Under this supposition our equation becomes

$$x^n - Px^{n-r} - Px^{n-r-1} \ldots\ldots - P = 0.$$

Suppose a to be greater than the greatest root, then

$$a^n - Pa^{n-r} - Pa^{n-r-1} \ldots\ldots - P = R; \text{ or}$$

$$a^n - P\left(\frac{a^{n-r+1} - 1}{a - 1}\right) = R,$$

$$\therefore\ a^n = P\left(\frac{a^{n-r+1}-1}{a-1}\right) + R = \frac{P\,a^{n-r+1}}{a-1} - \frac{P}{a-1} + R.$$

Suppose a to have been so taken that $\frac{P}{a-1} = R$, as may always be done; then we shall have

$$a^n = \frac{P\,a^{n-r+1}}{a-1},\ \text{or}\ (a-1)\,a^{r-1} = P.\quad \text{But}\ a-1 < a;$$

$$\therefore\ (a-1)^{r-1} < a^{r-1},\ (a-1)^r < (a-1)\,a^{r-1}.$$

Therefore, since $(a-1)\,a^{r-1} = P$, $(a-1)^r < P$;

$$\therefore\ a-1 < \sqrt[r]{P},\ \text{or}\ a < \sqrt[r]{P} + 1.$$

But $a > x$, $\therefore \sqrt[r]{P} + 1 > x$, as was to be proved.— E. B. Weaver, Normal School, Lancaster Co., Pa.

ANALYTICAL SOLUTIONS "OF THE TEN PROBLEMS IN THE TANGENCIES OF CIRCLES; AND ALSO OF THE FIFTEEN PROBLEMS IN THE TANGENCIES OF SPHERES."

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Part of the title of this article is derived from that of Major Alvord's paper in the eighth volume of the "*Smithsonian Contributions to Knowledge.*" The analytic solutions here given were obtained in the winter of 1856, soon after I became acquainted with Major Alvord's paper; they were attempted in consequence of the statements in that paper, that no mathematician had yet succeeded in reducing the analytical solution of these problems to equations of the second degree. At the end of his geometrical solutions, Major Alvord states, that the algebraic solution leads to an equation of the *eighth degree* when applied to the most general problem of the tangencies of circles; and to an equation of the *sixteenth degree* when applied to the most general problem of the tangencies of spheres. It will be found, however, that in this article both of these general

problems, which include all the others contained in the Smithsonian paper, are solved completely by equations not surpassing the second degree. The solution of the *particular* problems included in the two *general* problems here treated may be obtained either by applying to them the general equations developed in this article; or by treating them separately in a similar manner, the process being of still easier application to most of these particular problems. Thus the whole twenty-five problems of the Smithsonian paper may be considered as reduced to equations of the second degree, when the two most comprehensive problems have been so reduced.

First General Problem.

"To draw a Circle tangent to three given Circles."

Let the radii of the given circles be denoted by r_1, r_2, r_3, respectively, and let the corresponding co-ordinates of their centres be denoted by (a_1, b_1), (a_2, b_2), (a_3, b_3); also let ϱ, (α, β) be the radius and centre of the required circle, all being referred to any system of rectangular co-ordinate axes in the common plane of the circles. The equations of these circles will be respectively

(1.) $$(x - a_1)^2 + (y - b_1)^2 = r_1^2,$$

(2.) $$(x - a_2)^2 + (y - b_2)^2 = r_2^2,$$

(3.) $$(x - a_3)^2 + (y - b_3)^2 = r_3^2,$$

(4.) $$(x - \alpha)^2 + (y - \beta)^2 = \varrho^2,$$

Let x_1, y_1 be the point of contact of (1) and (4),
" x_2, y_2 " " " (2) and (4),
" x_3, y_3 " " " (3) and (4).

Then the point x_1, y_1 must satisfy the conditions,

(5.) $$(x_1 - a_1)^2 + (y_1 - b_1)^2 = r_1^2,$$

(6.) $$(x_1 - \alpha)^2 + (y_1 - \beta)^2 = \varrho^2,$$

(7.) $$\frac{d y_1}{d x_1} = -\frac{x_1 - a_1}{y_1 - b_1} = -\frac{x_1 - \alpha}{y_1 - \beta},$$

and similar conditions exist for the circles (2) and (3) combined with (4); but they may readily be derived from (5), (6), and (7) by simply changing the subscript 1 into 2 or 3.

From (7) we have

$$\frac{(x_1 - a_1)^2}{(y_1 - b_1)^2} + 1 = \frac{(x_1 - \alpha)^2}{(y_1 - \beta)^2} + 1,$$

$$\therefore \frac{r_1^2}{(y_1 - b_1)^2} = \frac{\varrho^2}{(y_1 - \beta)^2}.$$

(8.) $$\therefore y_1 - \beta = \pm \frac{\varrho}{r_1}(y_1 - b_1),$$

(9.) $$x_1 - \alpha = \pm \frac{\varrho}{r_1}(x_1 - a_1),$$

$$\therefore y_1 = \frac{r_1 \beta \mp b_1 \varrho}{r_1 \mp \varrho}, \qquad x_1 = \frac{r_1 \alpha \mp a_1 \varrho}{r_1 \mp \varrho},$$

$$\therefore y_1 - b_1 = \frac{r_1(\beta - b_1)}{r_1 \mp \varrho}, \qquad x_1 - a_1 = \frac{r_1(\alpha - a_1)}{r_1 \mp \varrho}.$$

Hence equation (5) becomes

$$(\alpha - a_1)^2 + (\beta - b_1)^2 = (r_1 \mp \varrho)^2,$$

(A.) and in like manner $(\alpha - a_2)^2 + (\beta - b_2)^2 = (r_2 \mp \varrho)^2,$

$$(\alpha - a_3)^2 + (\beta - b_3)^2 = (r_3 \mp \varrho)^2.$$

These equations evidently express the fact that the distance between the centres of any two of the tangent circles is equal to the sum or difference of their radii according as the contact is external or internal. It is also evident that the following are all the cases that can possibly occur, viz.: —

The required circle touches the circles (1), (2), (3) externally,

"	"	"	"	(1), (2), (3) internally,
"	"	"	"	(1), (2), ext. and (3) int.,
"	"	"	"	(1), (3), ext. and (2) int.,

The required circle touches the circles (2), (3), ext. and (1) int.,
" " " " (1), (2), int. and (3) ext.,
" " " " (1), (3), int. and (2) ext.,
" " " " (2), (3), int. and (1) ext.,

making eight cases, (the number found by Major ALVORD,) each of which is solved by a proper combination of the algebraic signs in the equations (A). In order to embrace all the cases in one solution, let $R_1 = \pm r_1$, $R_2 = \pm r_2$, $R_3 = \pm r_3$; then equations (A) may be written

$$\text{(A'.)} \qquad \begin{aligned} (\alpha - a_1)^2 + (\beta - b_1)^2 &= \{\pm (\varrho - R_1)\}^2 \\ (\alpha - a_2)^2 + (\beta - b_2)^2 &= \{\pm (\varrho - R_2)\}^2 \\ (\alpha - a_3)^2 + (\beta - b_3)^2 &= \{\pm (\varrho - R_3)\}^2. \end{aligned}$$

To find the values of the unknown quantities, α, β, ϱ, let

$$\text{(10.)} \qquad \alpha = A_1 \varrho + A_2, \qquad \beta = B_1 \varrho + B_2,$$

A_1, A_2, B_1, B_2 being functions of the *given* quantities, readily determined in each case by equations of the *first degree* in the following manner.

Developing the squares in the three equations (A′), and subtracting in turn the members of the first from those of the second and third, we have

$$\text{(11.)} \qquad \begin{aligned} 2(a_1-a_2)\alpha + 2(b_1-b_2)\beta &= 2(R_1-R_2)\varrho + a_1^2 - a_2^2 + b_1^2 - b_2^2 - (R_1^2 - R_2^2), \\ 2(a_1-a_3)\alpha + 2(b_1-b_3)\beta &= 2(R_1-R_3)\varrho + a_1^2 - a_3^2 + b_1^2 - b_3^2 - (R_1^2 - R_3^2). \end{aligned}$$

Substituting from (10), the values of α and β, equations (11) become

$$\begin{aligned} 2\,\{(a_1 - a_2)\,A_1 + (b_1 - b_2)\,B_1\}\,\varrho + 2\,(a_1 - a_2)\,A_2 + 2\,(b_1 - b_2)\,B_2 = \\ 2\,(R_1 - R_2)\,\varrho + a_1^2 - a_2^2 + b_1^2 - b_2^2 - (R_1^2 - R_2^2), \end{aligned}$$

$$\begin{aligned} 2\,\{(a_1 - a_3)\,A_1 + (b_1 - b_3)\,B_1\}\,\varrho + 2\,(a_1 - a_3)\,A_2 + 2\,(b_1 - b_3)\,B_2 = \\ 2\,(R_1 - R_3)\,\varrho + a_1^2 - a_3^2 + b_1^2 - b_3^2 - (R_1^2 - R_3^2). \end{aligned}$$

Now the two equations (11) are insufficient to determine the values of the three unknown quantities, α, β, ϱ; hence the result of the

preceding substitutions must be to give *identical equations*, or those capable of being verified by any value of ϱ whatever; hence the coefficients of the corresponding powers of ϱ must be equal.

Therefore

$$(12.)\quad \begin{aligned} (a_1 - a_2)\,A_1 + (b_1 - b_2)\,B_1 &= R_1 - R_2 \\ (a_1 - a_3)\,A_1 + (b_1 - b_3)\,B_1 &= R_1 - R_3, \end{aligned}$$

$$(13.)\quad \begin{aligned} (a_1-a_2)A_2+(b_1-b_2)B_2 &= \tfrac{1}{2}\{a_1^2-a_2^2+b_1^2-b_2^2-(R_1^2-R_2^2)\}, \\ (a_1-a_3)A_2+(b_1-b_3)B_2 &= \tfrac{1}{2}\{a_1^2-a_3^2+b_1^2-b_3^2-(R_1^2-R_3^2)\}. \end{aligned}$$

Hence

$$(14.)\quad \begin{aligned} A_1 &= \frac{(R_1 - R_2)\,(b_1 - b_3) - (R_1 - R_3)\,(b_1 - b_2)}{(a_1 - a_2)\,(b_1 - b_3) - (a_1 - a_3)\,(b_1 - b_2)} \\ B_1 &= \frac{(R_1 - R_3)\,(a_1 - a_2) - R_1 - R_2)\,(a_1 - a_3)}{(a_1 - a_2)\,(b_1 - b_3) - (a_1 - a_3)\,(b_1 - b_2)}, \end{aligned}$$

and by putting

$$\begin{aligned} D &= a_1^2 - a_2^2 + b_1^2 - b_2^2 - (R_1^2 - R_2^2) \\ E &= a_1^2 - a_3^2 + b_1^2 - b_3^2 - (R_1^2 - R_3^2) \\ F &= (a_1 - a_2)\,(b_1 - b_3) - (a_1 - a_3)\,(b_1 - b_2), \end{aligned}$$

we also get

$$(15.)\quad A_2 = \frac{(b_1 - b_3)\,D - (b_1 - b_2)\,E}{2\,F}, \quad B_2 = \frac{(a_1 - a_2)\,E - (a_1 - a_3)\,D}{2\,F}.$$

Thus the coefficients, A_1, B_1, A_2, B_2, of equations (10) are entirely determined in terms of the given quantities. From equations (10) we have

$$(16.)\quad \begin{aligned} \alpha - a_1 &= A_1\,(\varrho - R_1) + A_2 + A_1\,R_1 - a_1 = A_1\,(\varrho - R_1) + m \\ \beta - b_1 &= B_1\,(\varrho - R_1) + B_2 + B_1\,R_1 - b_1 = B_1\,(\varrho - R_1) + n, \end{aligned}$$

by making

$$(17.)\quad m = A_2 + A_1\,R_1 - a_1, \qquad n = R_2 + B_1\,R_1 - b_1.$$

Hence the first equation of (A′) becomes

$$(18.)\quad (\varrho - R_1)^2 - \frac{2\,(m\,A_1 + n\,B_1)}{1 - (A_1^2 + B_1^2)}\,(\varrho - R_1) = \frac{m^2 + n^2}{1 - (A_1^2 + B_1^2)}.$$

Hence

$$(19.)\qquad \varrho = R_1 + \frac{m\,A_1 + n\,B_1 \pm \sqrt{m^2 + n^2 - (n\,A_1 - m\,B_1)^2}}{1 - (A_1^2 + B_1^2)}.$$

Thus the equations (10), (14), (15), (17), and (19) completely solve the problem for every possible case.

It is evident that the problem is impossible whenever

$$(n\,A_1 - m\,B_1)^2 > m^2 + n^2;$$

and that of the two roots in (19), the one which gives a negative value for ϱ is to be interpreted as giving a circle of contact the reverse of that implied by the values assumed for R_1, R_2, and R_3. If, for example, the value assumed for R_2 implied *external* contact, then the negative value of ϱ would indicate *internal* contact with the second given circle.

I shall apply the preceding formulæ to determine the circle which passes through three given points, regarded as circles of infinitely small radius. Let the axis of abscissas pass through two of the given points, (a_2, b_2) and (a_3, b_3), and let the origin bisect the distance between them; hence $a_3 = -a_2$, $b_2 = b_3 = 0$; also, $r_1 = r_2 = r_3 = 0 = R_1 = R_2 = R_3$.

Hence $\qquad A_1 = 0, \quad B_1 = 0, \quad A_2 = 0,$

$$B_2 = \tfrac{1}{2}\frac{a_1^2 + b_1^2 - a_2^2}{b_1}; \quad m = -a_1, \quad n = \frac{a_1^2 - b_1^2 - a_2^2}{2\,b_1},$$

$$\varrho = \pm\sqrt{m^2 + n^2} = \pm\sqrt{\frac{4\,a_1^2\,b_1^2 + (a_1^2 - b_1^2 - a_2^2)^2}{4\,b_1^2}};$$

$$\alpha = 0, \beta = \tfrac{1}{2}\,\frac{a_1^2 - b_1^2 - a_2^2}{b_1}.$$

The remaining point, (a_1, b_1) is thus far subject to no restriction; it is anywhere on the plane of the co-ordinate axes. If now we suppose first that it is on the same straight line with the other two points, then $b_1 = 0$; hence $\beta = \infty$, $\varrho = \pm\infty$. Hence the radius

of the circle passing through three given points on the same straight line is infinite, and its centre is at an infinite distance from that line. The double sign of ϱ shows that the contact may be considered in this case either external or internal. Secondly, if we suppose the point (a_1, b_1) to be on the axis of ordinates, at a distance from the origin equal to a_2, then $a_1 = 0$, $b_1 = a_2$, $\beta = 0$, $\alpha = 0$, $m = 0$, $n = -a_2$, $\varrho = \pm a_2$, the circle having either external or internal contact with the three given points; and it is obvious that the results are correct.

As a second example, let it be required to find the circle which shall touch *externally* the three circles whose radii and centres are as follows:

$$\begin{array}{llll} r_1 = 1, & r_2 = 3, & r_3 = 2, & \text{hence } R_1 = -r_1 = -1, \\ a_1 = 0, & a_2 = 5, & a_3 = -3, & \quad " \quad R_2 = -r_2 = -3, \\ b_1 = 0, & b_2 = 0, & b_3 = 4, & \quad " \quad R_3 = -r_3 = -2. \end{array}$$

It will be found that $A_1 = -0.4$, $B_1 = -0.55$, $A_2 = 1.7$, $B_2 = 4.025$, $m = 2.1$, $n = 4.575$, $\varrho = 2.03672$, $\alpha = 0.885312$, $\beta = 2.904804$, and the equations (A), or (A'), will be found to be satisfied.

Second General Problem.

" To draw a Sphere tangent to four given Spheres."

Let the radii of the given spheres be denoted by r_1, r_2, r_3, r_4, and the co-ordinates of their centres by (a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3), (a_4, b_4, c_4); let ϱ $(\alpha, \beta, \gamma,)$ be the radius and centre of the required sphere; all being referred to rectangular co-ordinate axes in space.

It is evident that by an analysis similar to that for the circles, the distance of the centre $(\alpha, \beta, \gamma,)$ from each of the centres (a_1, b_1, c_1), &c., would be found equal to the sum or difference of the corresponding radii, according as the spheres had external or internal contact; hence to determine the four unknown quantities, $\alpha, \beta, \gamma, \varrho$, we should have the four equations

$$(C.)\qquad \begin{aligned}(\alpha - a_1)^2 + (\beta - b_1)^2 + (\gamma - c_1)^2 &= (\varrho \pm r_1)^2,\\ (\alpha - a_2)^2 + (\beta - b_2)^2 + (\gamma - c_2)^2 &= (\varrho \pm r_2)^2,\\ (\alpha - a_3)^2 + (\beta - b_3)^2 + (\gamma - c_3)^2 &= (\varrho \pm r_3)^2,\\ (\alpha - a_4)^2 + (\beta - b_4)^2 + (\gamma - c_4)^2 &= (\varrho \pm r_4)^2.\end{aligned}$$

Or if, to embrace all the cases in one solution, we put $R_1 = \pm r_1$, $R_2 = \pm r_2$, &c., the preceding equations may be written

$$(C'.)\qquad \begin{aligned}(\alpha - a_1)^2 + (\beta - b_1)^2 + (\gamma - c_1)^2 &= (\varrho - R_1)^2,\\ (\alpha - a_2)^2 + (\beta - b_2)^2 + (\gamma - c_2)^2 &= (\varrho - R_2)^2,\\ (\alpha - a_3)^2 + (\beta - b_3)^2 + (\gamma - c_3)^2 &= (\varrho - R_3)^2,\\ (\alpha - a_4)^2 + (\beta - b_4)^2 + (\gamma - c_4)^2 &= (\varrho - R_4)^2.\end{aligned}$$

Let

$$(1.)\qquad \alpha = A_1\,\varrho + A_2, \qquad \beta = B_1\,\varrho + B_2, \qquad \gamma = C_1\,\varrho + C_2.$$

Then, by an analysis precisely similar to that employed in the general problem of the contact of circles, the following equations will be found for determining the values of A_1, B_1, C_1, A_2, B_2, C_2.

For the sake of abbreviation, let

$$(2.)\qquad \begin{aligned}M_2 &= a_1^2 - a_2^2 + b_1^2 - b_2^2 + c_1^2 - c_2^2 - (R_1^2 - R_2^2),\\ M_3 &= a_1^2 - a_3^2 + b_1^2 - b_3^2 + c_1^2 - c_3^2 - (R_1^2 - R_3^2),\\ M_4 &= a_1^2 - a_4^2 + b_1^2 - b_4^2 + c_1^2 - c_4^2 - (R_1^2 - R_4^2);\end{aligned}$$

then

$$(3.)\qquad \begin{aligned}(a_1 - a_2)\,A_1 + (b_1 - b_2)\,B_1 + (c_1 - c_2)\,C_1 &= R_1 - R_2,\\ (a_1 - a_3)\,A_1 + (b_1 - b_3)\,B_1 + (c_1 - c_3)\,C_1 &= R_1 - R_3,\\ (a_1 - a_4)\,A_1 + (b_1 - b_4)\,B_1 + (c_1 - c_4)\,C_1 &= R_1 - R_4.\end{aligned}$$

$$(4.)\qquad \begin{aligned}(a_1 - a_2)\,A_2 + (b_1 - b_2)\,B_2 + (c_1 - c_2)\,C_2 &= \tfrac{1}{2}\,M_2,\\ (a_1 - a_3)\,A_2 + (b_1 - b_3)\,B_2 + (c_1 - c_3)\,C_2 &= \tfrac{1}{2}\,M_3,\\ (a_1 - a_4)\,A_2 + (b_1 - b_4)\,B_2 + (c_1 - c_4)\,C_2 &= \tfrac{1}{2}\,M_4.\end{aligned}$$

In order to abridge the expressions for A_1, A_2, &c., resulting from these equations, let

$$(5.)\quad \begin{aligned} P_a &= \frac{(b_1 - b_3)(c_1 - c_4) - (b_1 - b_4)(c_1 - c_3)}{(b_1 - b_2)(c_1 - c_3) - (b_1 - b_3)(c_1 - c_2)}, \\ Q_a &= \frac{(b_1 - b_4)(c_1 - c_2) - (b_1 - b_2)(c_1 - c_4)}{(b_1 - b_2)(c_1 - c_3) - (b_1 - b_3)(c_1 - c_2)}. \end{aligned}$$

Changing b into a, and a into b, in these expressions we have

$$(6.)\quad \begin{aligned} P_b &= \frac{(a_1 - a_3)(c_1 - c_4) - (a_1 - a_4)(c_1 - c_3)}{(a_1 - a_2)(c_1 - c_3) - (a_1 - a_3)(c_1 - c_2)}, \\ Q_b &= \frac{(a_1 - a_4)(c_1 - c_2) - (a_1 - a_2)(c_1 - c_4)}{(a_1 - a_2)(c_1 - c_3) - (a_1 - a_3)(c_1 - c_2)}. \end{aligned}$$

Changing again c into b, and b into c, in the last formulæ, we have

$$(7.)\quad \begin{aligned} P_c &= \frac{(a_1 - a_3)(b_1 - b_4) - (a_1 - a_4)(b_1 - b_3)}{(a_1 - a_2)(b_1 - b_3) - (a_1 - a_3)(b_1 - b_2)}, \\ Q_c &= \frac{(a_1 - a_4)(b_1 - b_2) - (a_1 - a_2)(b_1 - b_4)}{(a_1 - a_2)(b_1 - b_3) - (a_1 - a_3)(b_1 - b_2)}. \end{aligned}$$

$$(8.)\quad \begin{aligned} \therefore\; A_1 &= \frac{(R_1 - R_2)\, P_a + (R_1 - R_3)\, Q_a + R_1 - R_4}{(a_1 - a_2)\, P_a + (a_1 - a_3)\, Q_a + a_1 - a_4}, \\ B_1 &= \frac{(R_1 - R_2)\, P_b + (R_1 - R_3)\, Q_b + R_1 - R_4}{(b_1 - b_2)\, P_b + (b_1 - b_3)\, Q_b + b_1 - b_4}, \\ C_1 &= \frac{(R_1 - R_2)\, P_c + (R_1 - R_3)\, Q_c + R_1 - R_4}{(c_1 - c_2)\, P_c + (c_1 - c_3)\, Q_c + c_1 - c_4}. \end{aligned}$$

$$(9.)\quad \begin{aligned} A_2 &= \tfrac{1}{2}\, \frac{M_2\, P_a + M_3\, Q_a + M_4}{(a - a_2)\, P_a + (a_1 - a_3)\, Q_a + a_1 - a_4}, \\ B_2 &= \tfrac{1}{2}\, \frac{M_2\, P_b + M_3\, Q_b + M_4}{(b_1 - b_2)\, P_b + (b_1 - b_3)\, Q_b + b_1 - b_4}, \\ C_2 &= \tfrac{1}{2}\, \frac{M_2\, P_c + M_3\, Q_c + M_4}{(c_1 - c_2)\, P_c + (c_1 - c_3)\, Q_c + c_1 - c_4}. \end{aligned}$$

Equations (1) may be written as follows:

$$(10.)\quad \begin{aligned} \alpha - a_1 &= A_1\, (\varrho - R_1) + k, \\ \beta - b_1 &= B_1\, (\varrho - R_1) + m, \\ \gamma - c_1 &= C_1\, (\varrho - R_1) + n, \end{aligned}$$

by making

$$
(11.)\qquad
\begin{aligned}
k &= A_1\, R_1 + A_2 - a_1,\\
m &= B_1\, R_1 + B_2 - b_1,\\
n &= C_1\, R_1 + C_2 - c_1.
\end{aligned}
$$

Then the first of equations (C′) becomes

$$(\varrho - R_1)^2 - \frac{2\,(k\,A_1 + m\,B_1 + n\,C_1)}{1 - (A_1^2 + B_1^2 + C_1^2)}\,(\varrho - R_1) = \frac{k^2 + m^2 + n^2}{1 - (A_1^2 + B_1^2 + C_1^2)}.$$

Hence, by putting

$$G = (m\,A_1 - k\,B_1)^2 + (n\,A_1 - k\,C_1)^2 + (n\,B_1 - m\,C_1)^2,$$

we get

$$(12.)\qquad \varrho = R_1 + \frac{k\,A_1 + m\,B_1 + n\,C_1 \pm \sqrt{k^2 + m^2 + n^2 - G}}{1 - (A_1^2 + B_1^2 + C_1^2)}.$$

Thus the problem is solved analytically by means of equations not transcending the *second degree.*

The number of cases of a sphere touching four given spheres may be determined as follows:

The required sphere may touch the four given spheres all externally, 1 case; or all internally, 1 case; or two internally, two externally, 6 cases; or three externally, one internally, 4 cases; or three internally, one externally, 4 cases; making sixteen cases, as found by Major Alvord, all included in equations (C′).

I must not omit to notice what would otherwise seem to be a defect in the preceding solutions. In both problems there seem to be exceptional cases to which the formulæ are inapplicable. For example, in the tangencies of circles, if $b_1 = b_2 = b_3$, then it appears that $A_1 = \frac{0}{0}$, $A_2 = \frac{0}{0}$, and it is impossible to determine ϱ, α, β, by the proposed method. The reason of this is obvious; for the assumed equations (10), in the first problem, were only employed because the *two equations* (11) contained *three* unknown quantities, α, β, ϱ, and were, therefore, *indeterminate.* But when $b_1 = b_2 = b_3$,

one of the unknown quantities, β, disappears from the equations (11); hence the assumed equations (10) are inapplicable, and, at the same time, they are not wanted; for α and ϱ are then determined by the two equations (11), each of the *first degree*, and then β is found from either of the equations (A′) of the *second degree*. The same considerations apply to the corresponding case of the second general problem, when, for example, $c_1 = c_2 = c_3 = c_4$, in which case $P_a = \frac{0}{0}$, $Q_a = \frac{0}{0}$, $P_b = \frac{0}{0}$, $Q_b = \frac{0}{0}$; hence A_1, B_1, A_2, and B_2 are indeterminate. But under these conditions the unknown quantity γ disappears from the three equations of the *first degree* obtained in the early part of the process; hence the remaining quantities, α, β, ϱ, are easily found, and thence γ from one of the equations (C′), of the *second degree*. Thus in every case the problems are brought within the resolution of equations not surpassing the second degree. I might now give various applications of the second problem which I have computed, but omit them for fear of occupying too much space in the Monthly unnecessarily.

INSTANCES OF NEARLY COMMENSURABLE PERIODS IN THE SOLAR SYSTEM.

By Daniel Kirkwood, Professor of Mathematics, Indiana University.

The following instances of nearly commensurable periods in the solar system have not, I believe, been previously noticed: —

For the 3d and 4th satellites of Jupiter we have

$$7\,P^{\text{III.}} - 3\,P^{\text{IV.}} = 22\,m\ 24\,s = \tfrac{1}{1071}\,P^{\text{IV.}}; \ldots\ldots (a).$$

For the Uranian System,

$$13\,P^{\text{II.}} - 4\,P^{\text{IV.}} = 26\,m\ 43\,s = \tfrac{1}{726}\,P^{\text{IV.}}; \ldots\ldots (b).$$

$$10\,P^{\text{III.}} - 21\,P^{\text{II.}} = 51\,m\ 6\,s = \tfrac{1}{245}\,P^{\text{III.}}; \ldots\ldots (c).$$

For the asteroids and major planets,

11 periods of Mars	= 5	periods of	Astræa.
1 period of Jupiter	= 3	"	Calliope.
3 periods "	= 10	"	Clio.
4 " "	= 11	"	Virginia.
6 " "	= 13	"	Doris.
8 " "	= 29	"	Ariadne.
9 " "	= 22	"	Polyhymnia.
1 period of Saturn	= 9	"	Ariadne.
2 periods "	= 13	"	Leda.

We have also,

8 periods of Jupiter	= 7	periods of	TUTTLE's Comet.
6 " Saturn	= 13	"	"
31 " "	= 12	"	HALLEY's Comet.
6 " Neptune	= 13	"	"

Remarks. — The periods used in (*a*) are taken from HERSCHEL's Outlines; those in (*b*) and (*c*) from the Proceedings of the American Association for the Advancement of Science, 1854, p. 55. The periods of the asteroids are taken from the Mathematical Monthly for February, 1859. The period of TUTTLE's Comet is adopted from GOULD's Astronomical Journal, No. 118. That of HALLEY's is the mean of the six periods from 1378 to 1835. The epochs at which Neptune must have great perturbing influence on the motion of HALLEY's Comet are separated by intervals of about 988 years. The last occurred in the early part of the fifteenth century; — the next will be about the beginning of the twenty-fifth. It may also be remarked that the planets Jupiter and Uranus present a case of almost exact commensurability. *Eighty-five times the period of the former, minus twelve times that of the latter,* forms a quantity which amounts to only $\frac{1}{1102}$ of the period of Uranus. Hence in about $1008\frac{1}{4}$ years from the epoch of any conjunction of these planets, another conjunction will occur only *twenty-three minutes* from the place of the former.

THE ELEMENTS OF QUATERNIONS.

[Continued from Page 101.]

IV. — Scalars and Vectors.

19. If $\angle q = \frac{\beta}{\alpha} = 0°$, or $= 180°$, then β has either the same direction as α or the opposite, and q degenerates into either a positive or negative real number, and is then called a *Scalar*. Like tensors, scalars may be applied to any line in space without regard to its direction; and like them, they are commutative in combination with any quaternion. The product of scalars is evidently a scalar; and the conjugate of a scalar does not differ from the scalar itself.

20. If $\angle q = \pm 90°$, then q is called a *Vector*. Vectors will be denoted by v, v', &c. It is evident from (4) and (3′) that

(16) $$\mathrm{K}v = -v.$$

If $\mathrm{T}v = m$, then by (11),

(17) $$v^2 = -m^2;$$

that is, *the square of a vector is a negative scalar.* The square root of (17), $v = m\sqrt{-1}$, may be regarded as the expression for a vector of *indeterminate axis,* whose tensor is m. When $m = 1$, $v = \sqrt{-1}$ is called a *unit vector.* $\sqrt{-1_q}$ may be used to denote this vector, when its axis is fixed in the direction of Ax. q. The geometrical signification then of $\sqrt{-1}$ in this system, is the operation of turning any line to which it may be applied through an angle of 90° around any axis perpendicular to that line. Any quaternion may be expressed as a power of a vector, since, by (11), if, m being a positive number,

(18) $$q = [m\sqrt{-1_q}]^n, \qquad \text{then } \mathrm{T}q = m^n, \qquad \angle q = n\,90°,$$

and the independent real quantities m and n are sufficient to determine $\mathrm{T}q$ and $\angle q$.

21. The lines α and β being given, let the directions of the lines γ and δ be respectively parallel and perpendicular to that of α, and

let their lengths be such that $\gamma + \delta = \beta$. These conditions completely determine γ and δ, both of which will be in the plane of α and β. Then, by § 8,

$$q = \beta \div \alpha = (\gamma + \delta) \div \alpha = \gamma \div \alpha + \delta \div \alpha;$$

but $\gamma \div \alpha$ is a scalar, and $\delta \div \alpha$ is a vector. Any given quaternion then may be resolved into a sum of two other determinate quaternions, one of which is a scalar, and the other a vector; in this case the former is called the *scalar of the given quaternion*, and the latter *its vector*. The scalar of a quaternion, q, is written $\mathrm{S}\,q$, and its vector $\mathrm{V}\,q$; thus

(19) $$q = \mathrm{S}q + \mathrm{V}q = \mathrm{V}q + \mathrm{S}q.$$

From § 19, (16), and (4) we obtain

(20) $$\mathrm{K\,S}q = \mathrm{S}q = \mathrm{S\,K}q, \qquad \mathrm{K\,V}q = -\mathrm{V}q = \mathrm{V\,K}q;$$ whence

(19 *a*) $$\mathrm{K}q = \mathrm{S}q - \mathrm{V}q.$$

The sum and difference of (19) and (19 *a*) gives

(21) $$q + \mathrm{K}q = 2\,\mathrm{S}q, \qquad \text{and } q - \mathrm{K}q = 2\,\mathrm{V}q;$$ or

symbolically, $\mathrm{S} = \frac{1}{2}(1 + \mathrm{K})$, and $\mathrm{V} = \frac{1}{2}(1 - \mathrm{K})$.

If two or more quaternions are equal, their scalars must be equal, and also their vectors; and conversely if their scalars and vectors are equal each to each, the quaternions must be equal.

22. Let the planes of the two vectors, v and v', intersect in the line α; and let $v\,\alpha = \beta$, and $v'\,\alpha = \beta'$; then both β and β' will be in the plane of Ax.v and Ax.v', and, if $\gamma = \beta + \beta'$, $(v + v') = \gamma \div \alpha$. But $\gamma \div \alpha$ *is a vector*, whose axis, in the plane of Ax.v and Ax.v', makes the same angles with these axes that the line γ does with the lines β and β' respectively, and whose tensor bears the same ratios to $\mathrm{T}\,v$ and $\mathrm{T}\,v'$ that $\mathrm{T}\,\gamma$ does to $\mathrm{T}\,\beta$ and $\mathrm{T}\,\beta'$ respectively. If then we set off on Ax.v and Ax.v' lines whose tensors are equal

respectively to $\mathrm{T}\,v$ and $\mathrm{T}\,v'$, a parallelogram formed on these two sides will be similar to the parallelogram formed on β and β', and its diagonal corresponding to the diagonal γ of the latter parallelogram, will be the axis of $(v+v')$, and the tensor of this diagonal will be equal to the tensor of $(v+v')$. Hereafter we shall consider *the axis of a vector as a line of definite length, whose tensor is equal to the tensor of the vector.* With this definition, then,

$$\Sigma\,\mathrm{Ax.}\,v = \mathrm{Ax.}\,\Sigma v. \tag{22}$$

23. Let the line β be the intersection of the planes of v' and v, and such that rotation around it is positive from $\mathrm{Ax.}\,v'$ to $\mathrm{Ax.}\,v$; and let $v' = \alpha \div \beta$, and $v = \beta \div \gamma$; then both α and γ will be in the plane of $\mathrm{Ax.}\,v'$ and $\mathrm{Ax.}\,v$, and if we put

$$v'\,v = \alpha \div \gamma = q;\ \text{then } \mathrm{Ax.}\,q = \beta, \text{ and } \angle q = 180^\circ - \frac{\mathrm{Ax.}\,v}{\mathrm{Ax.}\,v'}. \tag{23}$$

Since $\mathrm{U}\,v'\,v^{-1} = \mathrm{U}\,(\alpha \div \beta)\,\mathrm{U}\,(\beta \div -\gamma) = \mathrm{U}\,(\alpha \div -\gamma)$, if we put

$$v'\,v^{-1} = -\,\mathrm{T}\,v^{-2}\,.\,q = q';\ \text{then}$$

$$\mathrm{Ax.}\,q' = \beta, \quad \text{and } \angle q' = 180^\circ + \angle q = \frac{\mathrm{Ax.}\,v'}{\mathrm{Ax.}\,v}. \tag{23 a}$$

As a special case of (8), we have, by (16),

$$\mathrm{K}\,v'\,v = \mathrm{K}\,v\,.\,\mathrm{K}\,v' = v\,v'. \tag{24}$$

24. Let $\mathrm{Ax.}\,v$ be in the plane of q, and α a line in the direction of $\mathrm{Ax.}\,q$; and let $v\,\alpha = \beta$, and $q\,\beta = \gamma$. Then

$$q\,v = (\gamma \div \beta)\,(\beta \div \alpha) = \gamma \div \alpha, \qquad \textit{a vector}, \tag{25}$$

$\mathrm{T}\,q\,v = \mathrm{T}\,q\,.\,\mathrm{T}\,v$, and $\mathrm{Ax.}\,q\,v$ is so placed in the plane of q that $\frac{\mathrm{Ax.}\,q\,v}{\mathrm{Ax.}\,v} = \frac{\gamma}{\beta} = \angle q$. Therefore,

$$\mathrm{Ax.}\,q\,v = q\;\mathrm{Ax.}\,v. \tag{26}$$

From (25), (16), and (8) we deduce

$$\mathrm{K}\,q\,v = -\,q\,v = \mathrm{K}\,v\,.\,\mathrm{K}\,q = -\,v\,.\,\mathrm{K}\,q, \qquad q\,v = v\,.\,\mathrm{K}\,q; \tag{27}$$

and putting $\mathrm{K}q$ for q in (27), we get, by (5),

(27 *a*) $$\mathrm{K}q \,.\, v = v \,.\, \mathrm{K}^2 q = v\,q;$$

whence, making the same substitution in (26), we get by (27 *a*)

(26 *a*) $$\mathrm{Ax}.\, v\,q = \mathrm{K}q \,.\, \mathrm{Ax}.\, v.$$

Putting $q\,v = v'$, (26) becomes $\mathrm{Ax}.\, v' = q\,\mathrm{Ax}.\, v$; whence

(28) $q = v' \div v = v'\,v^{-1} = \mathrm{Ax}.\, v' \div \mathrm{Ax}.\, v$, or, putting v^{-1} for v,

(28 *a*) $$v'\,v = \mathrm{Ax}.\, v' \div \mathrm{Ax}.\, v^{-1}.$$

25. Let the line α^{-1}, the *reciprocal* of the line α, be defined by the equations

(29) $\alpha = \mathrm{Ax}.\, v$, $\alpha^{-1} = \mathrm{Ax}.\, v^{-1}$; whence $(\mathrm{Ax}.\, v)^{-1} = \mathrm{Ax}.\, v^{-1}$.

Equations (26) and (28 *a*) give respectively by the substitution of (29)

(30) $$\mathrm{Ax}.\, q\,v^{-1} = q\,(\mathrm{Ax}.\, v)^{-1},$$

(31) $$v'\,v = \mathrm{Ax}.\, v' \div (\mathrm{Ax}.\, v)^{-1} = \mathrm{Ax}.\, v' \,.\, \mathrm{Ax}.\, v.$$

26. The results expressed in (22), (28), (31), (26), and (30) show that a vector, v, is, under the definitions assumed, (in § 22 and (29),) perfectly represented in these cases by a line, $\mathrm{Ax}.\, v$. Thus in § 22 the sum of vectors is proved *to be a vector*, and (22) expresses that the axis of this vector may be found by adding the axes of the given vectors. Equation (28) shows that the quotient of two vectors is the same quaternion as the quotient of their axes. Equation (31) shows that the definition (29) is so taken as to represent the multiplication of two vectors by the multiplication of their axes. Lastly, (25) shows that the product $q\,v$ is, in that special case, *a vector*, and (26) shows that the axis of this vector may be found by multiplying $\mathrm{Ax}.\, v$ by q; while (30) shows that the same property is true of $q\,v^{-1}$. The only possible cases remaining are the forms $q + v$, $q\,v$ and $q\,v^{-1}$, (except the special cases (25) and (30),) $v\,q$ and $v^{-1}\,q$;

and since these have, as yet, no meaning when a line, as Ax. v, is put in the place of v, we may now give them an arbitrary meaning in accordance with the conclusion, deduced from the other cases, that Ax. v may *always* be substituted for v; that is, that *a line is a vector*, and that *whatever may be proved of lines is proved of the vectors of which they are the axes.* HAMILTON indeed introduces the term *vector* as a name for a straight line, deriving it from *vehere*, because a line is supposed *to carry* a point from one of its extremities to the other. Afterwards, assuming its other signification, as a special case of a quaternion, he proceeds somewhat in the reverse order to that in which the subject is developed here.

27. Any vector, v, may, by § 22, be expressed as the sum of three mutually rectangular vectors, v_x, v_y, and v_z, whose axes are equal in length to the projections* of Ax. v in their respective directions. If the tensors of v_x, &c., are denoted by $(\mathrm{T}\,v)_x$, &c., and their versors by i, j, and k respectively, we have

$$q = \mathrm{V}\,q + \mathrm{S}\,q = (\mathrm{T\,V}\,q)_x\, i + (\mathrm{T\,V}\,q)_y\, j + (\mathrm{T\,V}\,q)_z\, k + (\mathrm{S}\,q),$$

in which the four quantities in parentheses are four independent elements involved in the complete determination of q. *Compare* § 14.

28. The planes of the quaternions $q = \beta \div \alpha$ and $q' = \beta' \div \alpha$ intersect in the line α; and it is evident that $\mathrm{S}\,q\,.\,\alpha$ is the projection of β in the direction of α, and $\mathrm{V}\,q\,.\,\alpha$ its projection on a plane perpendicular to α; whence it follows for the two quaternions, q and q', and thence by easy extension for any number, that

$$(32) \qquad \mathrm{S}\,\Sigma = \Sigma\,\mathrm{S}, \qquad \text{and} \qquad \mathrm{V}\,\Sigma = \Sigma\,\mathrm{V}.$$

But, as the value of $\mathrm{S}\,\Pi$ does not depend on what or how many factors the product may be composed of, so long as the product

* For greater completeness we may add that the *projection* of a line in any given direction, that is, on any given line, is so much of the latter line as is intercepted between perpendiculars dropped upon it from the extremities of the former.

itself remains the same; while ΠS evidently does depend on the composition of Π; it follows that we do *not* have in general S $\Pi =$ ΠS. And as § 23 shows that ΠV is not necessarily a vector, we do *not* have in general V $\Pi = \Pi$V.

29. We are now prepared to deduce the following relations, connecting the notation of quaternions with trigonometry. If $q = \beta \div \alpha$,

(33) $\quad \mathrm{S}q\,.\,\mathrm{T}\alpha = \mathrm{T}\beta\,.\cos{}^{\beta}_{\alpha},\ \mathrm{S}q = (\mathrm{T}\beta \div \mathrm{T}\alpha)\cos{}^{\beta}_{\alpha} = \mathrm{T}q\,.\cos{}^{\beta}_{\alpha},$

(33a) $\mathrm{V}q.\mathrm{T}\alpha = \mathrm{TV}q.\mathrm{T}\alpha.\mathrm{UV}q = \mathrm{T}\beta.\sin{}^{\beta}_{\alpha}.\sqrt{-1}_q,\ \mathrm{V}q = \mathrm{T}q.\sin{}^{\beta}_{\alpha}.\sqrt{-1}_q;$

whence

(34) $\quad q = \mathrm{T}q(\cos{}^{\beta}_{\alpha} + \sin{}^{\beta}_{\alpha}.\sqrt{-1}_q), \qquad \mathrm{U}q = \cos{}^{\beta}_{\alpha} + \sin{}^{\beta}_{\alpha}.\sqrt{-1}_q,$

and

(35) $\quad (\mathrm{S}q)^2 - (\mathrm{V}q)^2 = \mathrm{T}q^2\left(\cos^2{}^{\beta}_{\alpha} + \sin^2{}^{\beta}_{\alpha}\right) = \mathrm{T}q^2 = (\mathrm{S}q)^2 + (\mathrm{TV}q)^2.$

The second of equations (34) is equivalent to the two equations

(34′) $\quad \mathrm{SU}q = \cos{}^{\beta}_{\alpha}, \qquad \mathrm{VU}q = \sin{}^{\beta}_{\alpha}\,.\sqrt{-1}_q.$

The latter of these is equivalent to the two equations

(34″) $\quad \mathrm{TVU}q = \pm\sin{}^{\beta}_{\alpha},$* $\quad \mathrm{UVU}q = \pm\sqrt{-1}_q.$

By the substitution of (34′) in (33) and (33 a), we obtain

(36) $\quad \mathrm{S}q = \mathrm{T}q\,.\,\mathrm{SU}q, \qquad \mathrm{V}q = \mathrm{T}q\,.\,\mathrm{VU}q.$

Finally, by means of (34), (11), and (34′), we get

(37) $\quad \mathrm{U}q^m = (\cos{}^{\beta}_{\alpha} + \sin{}^{\beta}_{\alpha}\,.\sqrt{-1}_q)^m = \cos m\,{}^{\beta}_{\alpha} + \sin m\,{}^{\beta}_{\alpha}\,.\sqrt{-1}_q,$

which coincides with the expression of De Moivre's *Theorem*.

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* Tensors being always positive, if ${}^{\beta}_{\alpha} > 180$, it would be necessary to use the lower signs.

[To be Continued.]

NOTES ON THE THEORY OF PROBABILITIES.

By Simon Newcomb, Cambridge, Mass.

[Continued from Page 335, Vol. I.]

15. The problem which we are about to consider is one of the most fruitful in the theory of probabilities, as out of it grow the theory of errors, the theory of chance distribution, the law of averages, and the estimation of the probability that an observed concurrence of events is the result of a law of nature.

To find the probability that an event of which the probability on a single trial is p *will happen* s *times on* n *trials.*

The probability that it will fail on every trial is $(1-p)^n$, $1-p$ being the probability that it will fail on any single trial.

The probability that it will happen on the first trial and fail on the $n-1$ following ones is $p(1-p)^{n-1}$. But as the single event is as likely to occur on the 2d, 3d, ... nth trial as on the first, the probability that it will occur just once is $np(1-p)^{n-1}$.

The probability that the event will occur on the first two trials and fail on the $n-2$ subsequent ones is $p^2(1-p)^{n-2}$. But the two events can equally occur on the (1, 3), (1, 4) . . . . . $(1, n)$, or the (2, 3), (2, 4), &c. trials; in fact there will be $\overset{2}{\underset{n}{C}}$ pairs of trials on which the two events can occur, so that $\overset{2}{\underset{n}{C}} p^2(1-p)^{n-2}$ is the probability that it will occur twice.

By a process of reasoning exactly like the last, we find the probability that it will occur s times to be

$$(1) \qquad P_s = \overset{s}{\underset{n}{C}} p^s (1-p)^{n-s},$$

which is the $(s+1)$st term in the development of the binomial $[(1-p)+p]^n$. The sum of the probabilities of all the possible results of the n trials is therefore 1, as it ought to be.

As an example to elucidate the above, suppose that a cent is so formed that a head is twice as likely to be thrown as a tail, so that the probability of the former on each throw is $\frac{2}{3}$. If the coin is thrown four times, the results of the four throws may be as follows. After each separate result is written the fraction expressing the probability of that result.

No heads, $t\,t\,t\,t\ \frac{1}{3^4} = \frac{1}{81}$,
1 head, $h\,t\,t\,t$, $t\,h\,t\,t$, $t\,t\,h\,t$, $t\,t\,t\,h$, each $\frac{2}{81}$; $\times\ 4 = \frac{8}{81}$,
2 heads, $h\,h\,t\,t$, $h\,t\,h\,t$, $h\,t\,t\,h$, $t\,h\,h\,t$, $t\,h\,t\,h$, $t\,t\,h\,h$, $6 \times \frac{4}{81} = \frac{24}{81}$,
3 heads, $h\,h\,h\,t$, $h\,h\,t\,h$, $h\,t\,h\,h$, $t\,h\,h\,h$, $= \frac{32}{81}$,
4 heads, $h\,h\,h\,h = \frac{16}{81}$.

If we supposed heads as likely to be thrown as tails, we should find these probabilities to be $\frac{1}{16}$, $\frac{4}{16}$, $\frac{6}{16}$, $\frac{4}{16}$, $\frac{1}{16}$, respectively. The result would evidently be the same if we supposed that four coins were thrown from a box together.

16. To resume the general discussion, let us see what value of s is the most probable. This value we will determine by the condition that its probability must be greater than that of the next smaller number, and also greater than that of the next greater number, or

$$\overset{s}{\underset{n}{C}} p^s (1 - p)^{n-s} > \overset{s-1}{\underset{n}{C}} p^{s-1} (1 - p)^{n-s+1};$$
$$\overset{s}{\underset{n}{C}} p^s (1 - p)^{n-s} > \overset{s+1}{\underset{n}{C}} p^{s+1} (1 - p)^{n-s-1}.$$

Since $$\overset{s}{\underset{n}{C}} = \frac{n - s + 1}{s} \overset{s-1}{\underset{n}{C}}; \qquad \overset{s+1}{\underset{n}{C}} = \frac{n - s}{s + 1} \overset{s}{\underset{n}{C}};$$

we have from the division of the first inequality by $\overset{s-1}{\underset{n}{C}} p^{s-1} (1-p)^{n-s}$,

$$1 - p < \frac{n - s + 1}{s} p, \qquad \text{which gives } s < p\,(n + 1);$$

and from the division of the second inequality by $\overset{s}{\underset{n}{C}} p^s (1 - p)^{n-s-1}$ we have

$$1 - p > \frac{n-s}{s+1} p, \qquad \text{which gives } s > p\,(n+1) - 1;$$

s is therefore the greatest whole number in $p\,(n+1)$. If s and n are very large numbers, we have very nearly

$$(2) \qquad \frac{s}{n} = p.$$

It follows, therefore, that in a great number of trials events are more likely to occur a number of times proportional to their respective probabilities than any other number. Thus if a cent is thrown one hundred times, heads are more likely to be thrown fifty than any other single number of times. But it must not be supposed that they are therefore more likely to be thrown fifty times than not, for it is *almost* as likely to be thrown 49 times or 51 times as fifty times. The chances that it would be thrown *exactly* fifty times would be quite small, because there are so many other numbers that might be thrown.

17. Another deduction from the expression (1) is the following: however small the probability of an event on a single trial, by increasing the number of trials we can render the probability that the event will occur at least once as great as we please. For the probability that it will fail on every one of the n trials being $(1-p)^n$; however small p may be, we may make n so great that $(1-p)^n$ shall be as small as we please.

18. Suppose now that n is infinitely great, and p infinitely small, and that $np = \alpha$, α being a finite quantity. We may then put $n = n - 1 = n - 2$, &c. We shall then have, while s is finite,

$$\overset{s}{\underset{n}{C}} p^s = \frac{n^s p^s}{s!} = \frac{\alpha^s}{s!},$$

$$(1-p)^{n-s} = (1-p)^n = (1-p)^{\frac{\alpha}{p}} = e^{-\alpha},$$

e being the Napierian base. Substituting these values in (1) we obtain for the probability that the event will occur s times

(3) $$P_s = \frac{\alpha^s e^{-\alpha}}{s!}.$$

The probability that the event will fail, is therefore $e^{-\alpha}$; that it will occur once only, $\alpha e^{-\alpha}$; twice only, $\frac{\alpha^2}{2} e^{-\alpha}$, &c.

The sum of this series of probabilities continued to infinity is

$e^{-\alpha} (1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \text{\&c.,}) = e^{-\alpha} . e^{\alpha} = 1$, as it ought to be.

19. We may apply this equation to the determination of the probability that, if the stars were scattered at random over the heavens, any small space selected at random would contain s stars. Let N be the whole number of stars, h the number of units of space in the heavens, then $\frac{N}{h}\, dh$ may be taken to represent the infinitely small probability that the infinitely small space dh contains a star. Moreover, if l represents the extent of space selected at random which we consider, we may consider the examination of each dh as a trial, and the number n of trials will then be $\frac{l}{dh}$. The value of α will then become $\frac{N}{h} l$, and by substitution in (3) we have

(4) $$P = \frac{N^s l^s e^{-\frac{Nl}{h}}}{h^s s!}$$

for the probability that the space l contains s stars. Suppose, as a numerical example, that l is a square degree, and $N = 1500$, which is about the number of stars of the fifth and higher magnitudes; $s = 6$. We then have $\frac{l}{h} = \frac{1}{41253}$; by the substitution of these values in (4) we shall have the probability that any square degree selected at random in the heavens contains six stars. Multiplying this probability by 41253, the number of square degrees in the whole heavens, we obtain the probability that, *if the heavens were divided at random into square degrees, some one of those square degrees would contain six stars.* This probability we find to be

$$\frac{1500^6}{41253^5 \,.\, 6\,!} \,.\, e^{-.03636} = .000000128.$$

This, however, is evidently rather smaller than the probability that six stars should be found so near together that a square degree could be fitted on so as to include them.

Mr. Mitchell committed an error in his solution, the effect of which, if I mistake not, is to make his probability too great. His general method is, however, better applicable to this particular problem than that given above, but as there is a margin of vagueness and uncertainty about the problem in question, so that the answer does not admit of being expressed in exact numbers without an excessively complicated process of reasoning, I have preferred to deduce an approximate solution from the general formulæ, to be used in so many more problems.

20. Let us now consider Prof. Forbes's objections to the above results of the calculus of probabilities.

He scattered paint from a brush upon a wall, and found double and triple spots and groups innumerable. This is about as decisive as an attempt to disprove the Pythagorean proposition by measuring the squares described on a triangle without knowing whether it had or had not a right angle, and finding that one square was not equal to the sum of the others. As a mathematician would answer this objection by saying that his result only proved that he had either made a mistake in his measurements, or had not measured a right triangle; so Prof. Forbes's result only proves that either he was mistaken as to the marked character of the groupings, or that the proximity of the components of each group was the effect of their positions being determined *by the action of the same cause*, which is all that the theory of probabilities claims for the Pleiades. The latter supposition is by no means improbable, because a group of spots might be formed by the breaking up of a drop after it had left the brush.

21. Prof. Forbes remarked that an exactly uniform distribution of stars would not be expected as the result of a random distribution. In this he is correct; and as it is an interesting problem, we shall here determine what law a random distribution may be expected to follow. It may appear paradoxical to assert that the results of chance can be expected to follow any law; but such is really the case, and the formula (3) determines the law. As an example, suppose that the heavens are divided into 1500 equal portions, and that 1500 stars are distributed at random, or, to speak with more philosophical accuracy, that the causes which determine the position of each separate star are entirely independent of those which determine the position of any other. Then by reasoning as in § 19 we find $\alpha = 1$; and by formula (3) the probability that a unit of space selected at will contains no star, will be $\frac{1}{e} = \frac{1}{2.718\ldots}$*; one star, $\frac{1}{e}$; two stars, $\frac{1}{2e}$; three stars, $\frac{1}{2.3e}$; &c. If we then select the whole 1500 units we ought to expect the number which would be found to contain the several numbers of stars to be somewhere near 1500 multiplied by the respective probabilities, or

about $\frac{1500}{e}$	$= 552$	portions	containing	no star,
" $\frac{1500}{e}$	$= 552$	"	"	1 star,
" $\frac{1500}{2e}$	$= 276$	"	"	2 stars,
" $\frac{1500}{2.3e}$	$= 92$	"	"	3 stars,
" $\frac{1500}{4!e}$	$= 23$	"	"	4 stars,
" $\frac{1500}{5!e}$	$= 4(+)$	"	"	5 stars,
" $\frac{1500}{6!e}$	$= 1$	"	"	6 stars,

* The acute reader will perceive that the solutions in §§ 19 and 21 are those of a problem slightly (though not materially) different from that actually propounded.

and it would be quite improbable (about 1 chance to 8) that any space would be found to contain more than six stars.

If any one wishes an experimental illustration of the principle let him take a pint of rice, color a hundred grains of it black, mix the black grains thoroughly with the remainder, and stir the mixture till he finds six or eight of the black grains to form a group by themselves. Before this result arrives he will in all probability be willing to admit that should he ever see such a group in such a mixture, he would not believe that it was formed by indiscriminate mixing.

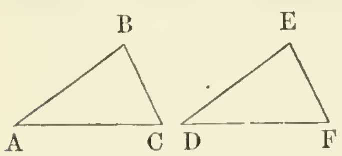
A SECOND BOOK IN GEOMETRY.

[Continued from Page 104.]

CHAPTER VI.

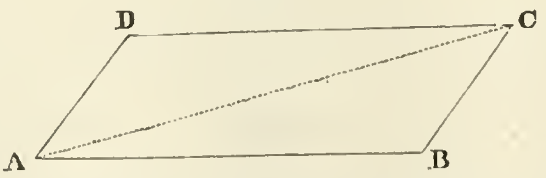
THE PYTHAGOREAN PROPOSITION.

91. *Theorem.* If a triangle has one side and the adjacent angles equal respectively to a side and the adjacent angles in another triangle, the two triangles are equal. — *Proof.* Let us suppose that, in the triangles A B C and D E F, we have the side A B equal to the side D E, the angle at A equal to the angle D, and that at B equal to that at E. Let us imagine the triangle D E F to be laid upon A B C in such a manner as to place E upon B, and D upon A, which can be done because A B is equal to D E. Now, as the angle A is equal to D, the line D F will run in the same direction as A C, and, as it starts from the same point, will coincide with it. Also, since the angle B is equal to E, the line E F will coincide with B C. Whence, by article 90, the triangles are equal.



92. *Theorem.* The opposite sides of a parallelogram are equal. — *Proof.* Article 90 gives us the only test of geometrical equality. So that in order to prove this theorem we must show that in a parallelogram like A B C D, A B may be made to coincide with D C, and B C with A D. And this would evidently be done if we could show that the triangle A B C is equal to A D C. But in these triangles the line A C is the same, and by article 87 the adjacent angles A C B and C A B are equal to the adjacent angles C A D and A C D; whence, by article 91, the two triangles are equal, and A D is equal to B C, and A B equal to D C.

Fig. B



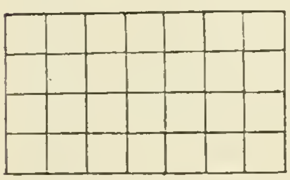
93. *Axiom.* If one end of a straight line stands still while the other turns round, the end that moves will *begin* to move in a direction at right angles to that of the line itself. Thus if A B were to begin to turn about the point A, B would *begin* to move either towards C or towards D. [If this proposition is not acknowledged as an axiom, the proof is in Select Propositions, Nos. 12 – 16.]

94. *Theorem.* The angles of a triangle cannot be altered without altering the length of the sides. — *Proof.* If in any triangle, as A B C, the sides were unchangeable, any alteration of the angles A and B would by article 93 make the point C move in two directions at once, (namely, at right angles to A C, and at right angles to B C,) which is impossible, and therefore the angles cannot be altered.

95. *Corollary.* If the three sides of a triangle are respectively equal to the three sides of another triangle, the angles of one must be equal to those of the other, and the equal angles are enclosed in the equal sides.

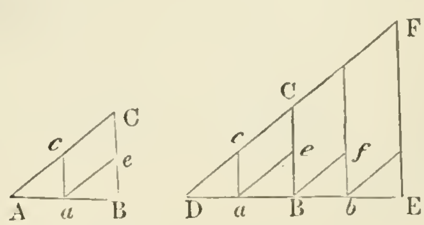
96. *Theorem.* If the opposite sides of a quadrangle are equal, the quadrangle is a parallelogram. — *Proof.* If in the quadrangle A B C D (Fig. B.), the sides A B and C D are equal, and also the sides A D and B C are equal, then, by drawing the diagonal A C, we have the triangles A B C and A D C, composed of equal sides, and, by article 95, the angle D A C must be equal to the angle A C B, and the angle D C A to the angle B A C; whence, by article 89, the figure is a parallelogram.

97. *Theorem.* The area of a rectangle is the product of its length by its breadth. — *Proof.* By drawing lines, at a distance apart equal to the unit of length, parallel to the sides of the rectangle, we shall (arts. 87 – 89) divide the rectangle into little squares, each of which is a unit of surface. Moreover, these squares are arranged in as many rows as there are units of length in one side of the rectangle, each row containing as many squares as there are units of length in the other side; so that the whole number of squares is found by multiplying the length of the rectangle by its breadth.



98. *Scholium.* In the above proof it is taken for granted that the sides of the rectangle can be divided into units of length. This can usually be done by taking the units sufficiently short, that is to say, if the lines are not an even number of inches in length, we may take tenths of an inch as the unit; if they are not even tenths, we can divide them into hundredths, or thousandths, or even millionths, of an inch. If after dividing each line into millionths of an inch anything less than the millionth of an inch were left at either end, it would be too small to be taken into consideration. There would be no error, even in reasoning, from neglecting it. For as long as anything is left at the ends of the lines, we can choose smaller units, but as long as the units are of any size at all, our reasoning holds good, and the rectangle is measured by the product of its dimensions.

99. *Theorem.* If the angles of one triangle are equal to those of another triangle, any two sides of one of the triangles have the same ratio to each other as that of the two sides including the same angle in the other triangle. — *Proof.* Let the triangles A B C and D E F be equiangular with respect to each other. Place the vertex A upon the vertex D, and allow the side A B to fall upon the side D E. Since the angles A and D are equal, the line A C will fall upon the line D F, and since the angles C and F are equal, the line B C will lie parallel to the line E F.



Let the sides A B and D E be divided into units of length, A *a*, *a* B, B *b*, &c. Through the

points of division draw lines $a c$, $b d$, &c., parallel to E F. Draw also the lines $a e$, Bf, &c.. parallel to D F. By article 91, the triangles A a c, a B e, &c. are equal. By article 92, $a e$ is equal to c C, Bf to C d, &c. Hence it is easy to see that A B is composed of the same number of times A a, that A C is of A c, and that in like manner D E is as many times D a, as D F is of D c. And thus by article 78, D E : D F = A B : D C, because each of these ratios is equal to B a : a e.

100. *Scholium.* If the lines A B and D E do not consist of a certain number of times the first unit of length which we have chosen, we may choose a unit small enough to make the remainder small enough to be neglected.

101. *Definitions.* The right angle, right triangle, legs, and hypotenuse, are defined in articles 14 and 17.

102. *Theorem.* The sum of the three angles of a triangle is equivalent to two right angles.

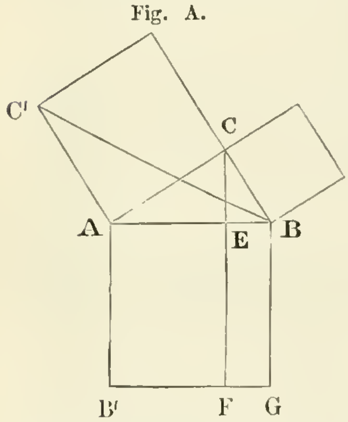
This proposition has been proved in articles 26 – 31, 34 – 36, and 57 – 62.

103. *Corollary.* The sum of the two angles opposite to the legs of a right triangle is equivalent to one right angle.

104. *Corollary.* If an angle opposite a leg in one right triangle is equal to an angle in another right triangle, the two right triangles are equiangular with respect to each other.

105. *Theorem.* If from the vertex of the right angle in a right triangle, a line be drawn at right angles to the hypotenuse dividing the hypotenuse into two segments, each leg is a mean proportional between the whole hypotenuse and the segment nearest the leg. — *Proof.* Let A B C be a right triangle with a right angle at C. Draw C F at right angles to A B. The triangle B E C is right angled at E, and has an angle at B equal that at B in the triangle A B C. Hence, by article 104, the triangle B E C has its angles equal to those of A B C. Hence, by article 99, B E : B C = B C : B A. In the same way A E : A C : : A C : A B.

Fig. A.



106. *Theorem.* The square on the hypotenuse is equivalent to the sum of the squares on the legs. — *Proof.* Let A C B (Fig. A.) be a right triangle, with a right angle at C, and let a square be drawn on each side. Draw C F at right angles to A B. The figure B F will be a rectangle, because all its angles will be right angles. It will, therefore, be measured by the product of B E into E F; or (since E F = B G and B G = B A), by the product of B E × B A. But since B C is a mean proportional between the lines B E and B A, this product is equal to B C × B C, which is the measure of the square on B C. That is, the measure of the rectangle B F is the same as that of the square on B C. In the same manner it may be shown that the rectangle A F is equivalent to the square on A C. But the sum of these two rectangles is evidently equal to the square on the hypotenuse.

107. In these thirty-one articles I have given you a proof of the Pythagorean proposition in the usual synthetic form. Parts of the proof are not completely filled out, but the omitted steps are so short and easy, that I think you will have no difficulty whatever in supplying them. Do not be satisfied with understanding each of the thirty-one articles, but examine them closely from the 76th to the 106th, and see whether I have introduced anything which is not necessary to the proof of 106. In making this examination, it will be most convenient for you to proceed backward.

These thirty-one articles have been here introduced as lemmas, i. e. preparatory propositions, for demonstrating the Pythagorean proposition. But they are also each one truths worth knowing, and will aid in establishing many theorems that have no connection with a right triangle.

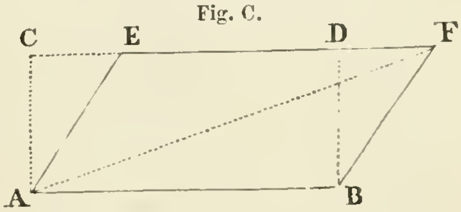
108. Another mode of analyzing this proposition would be suggested by our knowledge of the fact that any triangle is equivalent to half a rectangle of the same base and altitude. I will not lead you through this analysis, but will simply build up for you by synthesis, a

Second Proof of the Pythagorean Proposition.

109. *Definitions.* Any side of a triangle or quadrangle may be called its base, and the altitude of the figure is the distance from the base to the most distant vertex of the figure. This distance is measured by a straight line at right angles to the base, and contained between the vertex and the base, prolonged if need be.

110. *Theorem.* Every parallelogram is equivalent to a rectangle of the same base and altitude. — *Proof.* Let A B C D be a rectangle, and A B E F a parallelogram having the same base A B, and the same altitude B D. It is manifest that if the triangle B D F by which the parallelogram overlaps the rectangle is equal to the triangle A E C by which the rectangle overlaps the parallelogram, the two quadrangles are equivalent. But A E and its adjacent angles are equal to B F and its adjacent angles, and therefore the triangles are equal (Art. 91), and the quadrangles equivalent.

Fig. C.



111. *Theorem.* Every triangle is equivalent to half a rectangle of the same base and altitude. — *Proof.* Let A F B be a triangle, and A B C D a rectangle having the same base, A B and the same altitude B D (Fig. C.). Continue C D to F and draw A E parallel to B F. The triangle A E F has its three sides equal to those of A B F; the triangles are, therefore, equal to each other (Art. 95); and each is equal to half the parallelogram A B E F, which is equivalent to the rectangle A B C D.

112. *Theorem.* The square on the hypotenuse is equivalent to the sum of the squares on the legs. — *Proof.* Having drawn the figure (Fig. A.), as for the former proof, draw the lines C′ B, and B′ C. The triangle A B′ C has the same base A B′, and the same altitude A E as the rectangle A F, and is equivalent to half that rectangle. The triangle A B C′ has the same base A C′ and the same altitude A C as the square C′ C, and is equivalent to half that square. So that if the triangles A B C′ and A B′ C are equal, the rectangle is equivalent to the square. But these triangles are equal, for if A B′ C were turned about the vertex A as on a pivot until the point C covered C′, then B′ would cover B, and the triangles would coincide. For A C would rotate through a right angle, and A B′ through a right angle; and A C = A C′, and A B = A B′.

113. This proof of the Pythagorean proposition is more strictly geometrical than the preceding, as it does not involve the idea of multiplying lines to measure areas. But you must remember that each is equally conclusive. I have here also omitted some of the shorter steps. You should not only be able to fill out these steps when the omission is pointed out to you, but also to discover the omission for yourselves. Take the proofs which I have written down and examine them step by step, asking at each step, is that strictly self-evident? Can it be questioned? Can it be divided into two steps? Is there need of proof? If so, has the proof been given in a previous article? It is only by such an earnest study of the book and of the subject that you can make the process of mathematical reasoning become a sure and pleasant road for you to the discovery of truth.

[To be continued.]

Editorial Items.

THE following students have sent us solutions of the Prize Problems in the October Number, Vol. II., of the Monthly.

MISS AMANDA BENNETT, Red Wing, Minnesota. Prob. I.
MISS HARRIET L. ENSIGN, Catskill Academy, N. Y. Probs. I., II., III., and IV.
MISS HARRIET S. HAZELTINE, Worcester, Mass. Probs. I., II., and IV.
OAKLEY H. SMITH, New Hampton Institution, Fairfax, Vt. Probs. I. and II.
D. Y. BINGHAM, Ellicotteville, N. Y. Prob. III.
SAMUEL S. EASTWOOD, High School, Saxonville, Mass. Probs. I. and II.
C. P. MARSTON, Public High School, Hartford, Ct. Probs. I. and II.
WILLIAM W. JOHNSON, Sophomore Class, Yale College, Ct. Probs. III. and IV., and first part of V.
BENJAMIN F. WEBBER, Wesleyan Seminary, Kent's Hill, Maine. Probs. I. and II.
ISAAC H. TURRELL, Drewersburg, Ind. Probs. I., II., III., and IV.
GEORGE D. HALE, Select School, Adams Centre, N. Y. Probs. I. and II.
F. E. HASTINGS, Literary Institution, Suffield, Ct. Prob. I.
G. W. BROWN and ISAAC FULLER, Collegiate Institution, Bowden, Ga. Each Probs. I., II.
WILLIAM C. CLEVELAND, Lawrence Scientific School, Cambridge, Mass. All but V.
WILLIAM C. HENCK, High School, Dedham, Mass. Probs. I. and II.
CHARLES B. BOUTELLE, Waterville Academy, Maine. Prob. III.
M. H. DOOLITTLE, Sophomore Class, Antioch College, Yellow Springs, Ohio. Probs. III., IV., and first part of V.
W. D. MERSHON, Sophomore Class, Princeton College, N. J. Prob. IV.
R. E. LEONARD, Ward School, No. 32, New York City. Probs. I. and II.
J. B. FOSSETT, New London Institute, Ct. Probs. I., II., and III.
SAMUEL I. BALDWIN, Chester Institute, Chester, N. J. Probs. I. and II.
S. W. BURNHAM, New York City. Probs. I. and II.
M. L. STREATOR, Mayslick Academy, Ky. Probs. I., II., and III.
G. H. TOWER, Classical High School, Petersham, Mass. Probs. I. and II.
PERRIN B. PAGE, Literary Institute, Nunda, N. Y. Probs. I. and II.
J. D. VAN BUREN, Rensselaer Polytechnic Institute, Troy, N. Y. Probs. III., IV., and V.
O. B. WHEELER, Sophomore Class, University of Michigan, Ann Arbor. Probs. III., IV., and V.
JOSEPH COCKE and AYLETT B. COLEMAN, Locust Grove Academy, Albemarle Co., Va. Each Probs. I. and II.
M. K. BOSWORTH, Sophomore Class, Marietta College, Ohio. Probs. III., IV., and V.
W. A. AKEN, New Wilmington, Pa. Probs. I. and II.

The following solutions, unfortunately, did not reach us in time.

WILSON BERRYMAN and OTHO E. MICHAELIS, Sophomore Class, N. Y. Free Academy, each answered all the questions.

H. TIEMAN, Baltimore, Md., answered all the questions but the second part of Prob. V.

PHILO HOLCOMB, Hughes High School, Cincinnati, answered Prob. IV.

THE

MATHEMATICAL MONTHLY.

Vol. II. . . . FEBRUARY, 1860. . . . No. V.

PRIZE PROBLEMS FOR STUDENTS.

I. Prove that the value of a proper fraction is increased, and an improper fraction diminished, by adding the same quantity to both terms of the fraction; and that the reverse is the case when the same quantity is subtracted from both terms of the fraction.

II. A common tangent is drawn to two circles which touch each other externally; if a circle be described on that part of the tangent which lies between the points of tangency as diameter, this circle will pass through the point of contact of the two circles, and will touch the line joining their centres.

III. Find the n quantities $x_1, x_2, \ldots\ldots x_n$, from the n equations,

$$\begin{aligned}
&x_1 + x_2 + x_3 \ldots\ldots + x_n = A, \\
&a_1\, x_1 + a_2\, x_2 + a_3\, x_3 \ldots\ldots + a_n\, x_n = 0, \\
&a_1^2\, x_1 + a_2^2\, x_2 + a_3^2\, x_3^2 \ldots\ldots + a_n^2\, x_n = 0, \\
&\quad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot \\
&a_1^{n-1}\, x_1 + a_2^{n-1}\, x_2 + a_3^{n-1}\, x_3 \ldots\ldots + a_n^{n-1}\, x_n = 0,
\end{aligned}$$

and obtain symmetrical expressions for x_1, x_2, &c.

IV. Prove that $\sin^n(\theta - \varphi)\sin\varphi$ is a maximum when

$$\sin(\theta - 2\varphi) = \frac{n-1}{n+1}\sin\theta\,;$$

θ being a given constant, and φ the variable.

V. The notation of Problem V. in the November No. being retained, prove that in the plane triangle

$$\frac{1}{\varrho'} + \frac{1}{\varrho''} + \frac{1}{\varrho'''} = \frac{1}{\varrho};$$

and in the spherical triangle,

$$\frac{1}{\tan\varrho'} + \frac{1}{\tan\varrho''} + \frac{1}{\tan\varrho'''} = \frac{1}{\tan\varrho} \, \frac{\cos(\varrho + \delta)\cos(\varrho - \delta)}{\cos^2 r \cos^2 \varrho};$$

$$\frac{\cos\delta'}{\cos\varrho'} + \frac{\cos\delta''}{\cos\varrho''} + \frac{\cos\delta'''}{\cos\varrho'''} = \frac{\cos\delta}{\cos\varrho}.$$

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. II., Vol. II.

The first Prize is awarded to George B. Hicks, Cleveland, Ohio.

The second Prize is awarded to William Hinchcliffe, Barre Plains, Mass.

The third Prize is not awarded.

Prize Solution of Problem III.

By George B. Hicks, Cleveland, Ohio.

Of all right-angled plane triangles having the same given hypothenuse, to find the one whose area is the greatest possible. To be solved by Algebra.

Since the hypothenuse is constant, the area is evidently a maximum when the perpendicular dropped from the right angle upon the hypothenuse is a maximum. Let $x\,y$ be the sides, c the hypothenuse, p the perpendicular, and d one of the segments of the hypothenuse. Then

$$p^2 + d^2 = x^2, \qquad p^2 + (c - d)^2 = y^2, \qquad x^2 + y^2 = c^2;$$

whence (1) $d^2 - c\,d = -p^2.$

$$\therefore d = \frac{c}{2} \pm \sqrt{\frac{c^2}{4} - p^2}.$$

But the greatest value p^2 can have, and d still be real, is $\frac{1}{4}c^2$; $\therefore p = \frac{1}{2}c$, $d = \frac{1}{2}c$, $\therefore$ the triangle is isosceles, and $x = y = c\sqrt{\frac{1}{2}}$. Otherwise, put $d = w + \frac{1}{2}c$.

$\therefore p^2 = cd - d^2 = \frac{1}{4}c^2 - w^2$, which is obviously a maximum when $w = 0$, $\therefore p = \frac{1}{2}c$ and $x = y$ as before.

Or, (1) may be written $p^2 = d(c - d)$. But $d(c - d)$ is the product of the segments of the hypothenuse, which we know to be a maximum when the segments are equal; the result already found.

Second Solution.

By William Hinchcliffe, Barre Plains, Mass.

Let ABC be right-angled at C, from which drop the perpendicular CD. Put $AB = 2a$, $AD = a + x$, $BD = a - x$. Now, $\frac{1}{2}AB \times CD =$ a maximum. But AB is given; therefore CD and hence $CD^2 =$ a maximum. But $CD^2 = AD \times BD = a^2 - x^2$ $=$ a maximum when $x = 0$. $\therefore AD = BD = CD$. $\therefore AC = BC = a\sqrt{2}$; whence the required triangle will contain the greatest area when the sides containing the right angle are equal.

Prize Solution of Problem IV.

By William Hinchcliffe, Barre Plains, Mass.

What is that fraction, the cube of which being subtracted from it, the remainder is the greatest possible? To be solved by Algebra.

Let x denote the required fraction. Then, by the question, $x - x^3 = y =$ a maximum. Denote a negative root of this equation by $-a$; then $x^3 - x + y = 0$ divided by $x + a = 0$ gives a quotient, $x^2 - ax + a^2 - 1 = 0$ (1), and a remainder, $a^3 - a - y = 0$. But by solving, (1) gives

$$x = \frac{a \pm \sqrt{4 - 3a^2}}{2}.$$

Now, it is evident that, if y is positive, x is less than 1; and since $x - x^3 = y = a^3 - a$, it also follows, that when y is the greatest

possible, a must have the greatest value which will make x real.

$$\therefore\ 4 - 3a^2 = 0;\ \text{or}\ a = \frac{2}{\sqrt{3}}. \qquad \therefore\ x = \frac{1}{\sqrt{3}}.$$

Prize Solution of Problem V.

By George B. Hicks, Cleveland, Ohio.

If, in a plane or spherical triangle, A, B, C denote the angles, and a, b, c the sides respectively opposite them; and if we produce the sides of the triangle, and consider the three circles which touch two of the sides interiorly and the third side exteriorly; and denote by r, ϱ the radii of the circumscribed and inscribed circles; by $\varrho', \varrho'', \varrho'''$, the radii of the circles touching exteriorly the sides a, b, c respectively; by $\delta', \delta'', \delta'''$, the distances of the centres of these circles from the centre of the circle circumscribed about the primitive triangle; then we have in the plane

$$-\sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2} A \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C,$$
$$\sin A - \sin B + \sin C = 4 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} B \sin \tfrac{1}{2} C,$$
$$\sin A + \sin B - \sin C = 4 \sin \tfrac{1}{2} A \sin \tfrac{1}{2} B \cos \tfrac{1}{2} C,$$

and in the sphere

$$-\sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2} A \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C \cdot \frac{\cos \delta'}{\cos r \cos \rho'},$$
$$\sin A - \sin B + \sin C = 4 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} B \sin \tfrac{1}{2} C \cdot \frac{\cos \delta''}{\cos r \cos \rho''},$$
$$\sin A + \sin B - \sin C = 4 \sin \tfrac{1}{2} A \sin \tfrac{1}{2} B \cos \tfrac{1}{2} C \cdot \frac{\delta'''}{\cos r \cos \rho'''}.$$

First Part.— In plane trigonometry

$$\sin \tfrac{1}{2} A = \sqrt{\frac{(s-c)(s-b)}{bc}}, \qquad \cos \tfrac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}};$$

and similar expressions may be found for the other angles by simply permuting the letters. Now

$$-\sin A + \sin B + \sin C = 2(-\sin \tfrac{1}{2} A \cos \tfrac{1}{2} A + \sin \tfrac{1}{2} B \cos \tfrac{1}{2} B + \sin \tfrac{1}{2} C \cos \tfrac{1}{2} C)$$
$$= 2M\left(\frac{-a+b+c}{abc}\right),$$

putting $M = \sqrt{s(s-a)(s-b)(s-c)}$. But

$$4 \cos \tfrac{1}{2} A \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C = 2M\left(\frac{-a+b+c}{abc}\right),$$

observing that $s = \tfrac{1}{2}(a+b+c)$.

$$\therefore\ -\sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2} A \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C;$$

and in the same way the remaining equations may be found.

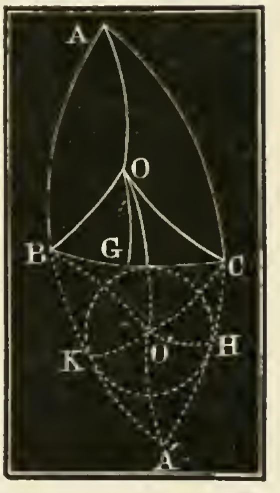
Second Part. — By a process precisely similar to the above we shall find for the spherical triangle

$$-\sin A + \sin B + \sin C = 2\, M \left(\frac{-\sin a + \sin b + \sin c}{\sin a \sin b \sin c}\right),$$

$$4 \cos \tfrac{1}{2} A \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C = 4\, M \left(\frac{\sin (s-a)}{\sin a \sin b \sin c}\right).$$

It only remains to show that

$$\frac{\cos \delta'}{\cos r \cos \varrho'} = \frac{-\sin a + \sin b + \sin c}{2 \sin (s-a)}.$$



Let $A\,B\,C$ be any spherical triangle, and O the centre of its circumscribed circle. Produce $A\,B$ and $A\,C$ until they meet in A'. Let $A'\,C = b_1$, and $A'\,B = c_1$. Put $B\,K = \alpha$, $K\,A' = \beta$, $C\,H = \gamma$. Then $a = \alpha + \gamma$, $b_1 = \beta + \gamma$, $c_1 = \alpha + \beta$.

$$\therefore\ B\,K = \alpha = \frac{a - b_1 + c_1}{2} = \frac{a - (\pi - b) + (\pi - c)}{2}$$

$$= \frac{a + b - c}{2} = s - c.$$

We also have $B\,O = r$, $O'\,K = \varrho'$, $O\,O' = \delta'$; and since $A\,O\,B$, $A\,O\,C$, $B\,O\,C$ are isosceles, $O\,B\,C = \frac{1}{2}(-A + B + C)$, $C\,B\,O' = \frac{1}{2}(\pi - B)$;

$$\therefore\ O\,B\,O' = \tfrac{1}{2}\pi - \tfrac{1}{2}(A - C). \quad \therefore\ \cos O\,B\,O' = \sin \tfrac{1}{2}(A - C) = \frac{\cos \delta' - \cos r \cos B\,O'}{\sin r \sin B\,O'}.$$

But from the right-angled triangle $B\,O'\,K$,

$$\cos B\,O' = \cos \varrho' \cos (s - c), \quad \sin B\,O' = \frac{\sin \varrho'}{\sin \frac{1}{2}(\pi - B)} = \frac{\sin \varrho'}{\cos \frac{1}{2} B}.$$

Substituting the values of $\cos B\,O'$, $\sin B\,O'$, in that of $\sin \frac{1}{2}(A - C)$, we get, by observing that $\dfrac{\sin \frac{1}{2}(A - C)}{\cos \frac{1}{2} B} = \dfrac{\sin \frac{1}{2}(a - c)}{\sin \frac{1}{2} b}$,

$$\frac{\cos \delta'}{\cos r \cos \varrho'} = \frac{\sin \frac{1}{2}(a - c)}{\sin \frac{1}{2} b} \tan r \tan \varrho' + \cos (s - c).$$

But by YOUNG'S Trig. pp. 128, 134,

$$\tan r \tan \varrho' = \frac{2 \sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c}{\sin (s - a)};$$

$$\therefore \frac{\cos \delta'}{\cos r \cos \varrho'} = \frac{2 \sin \frac{1}{2} a \sin \frac{1}{2} c \sin \frac{1}{2} (a - c) + \sin (s - a) \cos (s - c)}{\sin (s - a)}$$

$$= \frac{- \sin a + \sin b + \sin c}{2 \sin (s - a)}, \text{ as required.}$$

The value of $\tan r$ and $\tan \varrho'$ is easily found from the triangles $B\,O\,G$, $B\,O'\,K$, and $\tan r$ may be reduced to an expression in terms of the sides by making use of the usual formulas, which give $\sin \frac{1}{2} A$, $\cos \frac{1}{2} A$, &c. in terms of the sides, and $\sin \frac{1}{2} a$, $\cos \frac{1}{2} a$, &c. in terms of the angles. Whence will result the value of $\tan r \tan \varrho'$ given above.

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PRIZE ESSAY ON CENTRAL FORCES.

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1. In the following essay I shall take the well-known and fundamental equations of motion

$$(1) \quad D_t^2 x + X = 0, \quad D_t^2 y + Y = 0, \quad D_t^2 z + Z = 0,$$

for granted, as they are to be found in treatises on Mechanics. In these equations x, y, z are the rectangular co-ordinates of the moving body, and X, Y, Z are the forces directed along the co-ordinate axes of x, y, z respectively.

2. Let R represent the central force acting on the moving body, M; let α, β, γ be the angles which its direction makes with the co-ordinate axes x, y, z respectively; let r equal the distance which

separates the centre of gravity of the body M from the centre of gravity of the system, this latter point being the centre of attraction. We then have, assuming this centre for the origin of co-ordinates,

(2) $$X = R \cos\alpha, \quad Y = R \cos\beta, \quad Z = R \cos\gamma.$$

(3) $$x = r \cos\alpha, \quad y = r \cos\beta, \quad z = r \cos\gamma.$$

Equations (2) give

(4) $$\frac{X}{Y} = \frac{\cos\alpha}{\cos\beta}, \quad \frac{Y}{Z} = \frac{\cos\beta}{\cos\gamma}, \quad \frac{Z}{X} = \frac{\cos\gamma}{\cos\alpha}.$$

Equations (3) give

(5) $$\frac{x}{y} = \frac{\cos\alpha}{\cos\beta}, \quad \frac{y}{z} = \frac{\cos\beta}{\cos\gamma}, \quad \frac{z}{x} = \frac{\cos\gamma}{\cos\alpha}.$$

Substituting these values in equations (4), clearing of fractions, and transposing into the first member, we obtain

(6) $$Xy - Yx = 0, \quad Yz - Zy = 0, \quad Zx - Xz = 0.$$

Substituting in these equations the values of $X\,Y\,Z$, as given by equations (1), we obtain

(7) $$y\,D_t^2 x - x\,D_t^2 y = 0, \quad z\,D_t^2 y - y\,D_t^2 z = 0, \quad x\,D_t^2 z - z\,D_t^2 x = 0.$$

3. The integrals of (7) give, letting c, c', and c'' be the arbitrary constants introduced by integration,

(8) $$y\,D_t x - x\,D_t y = c, \quad z\,D_t y - y\,D_t z = c', \quad x\,D_t z - z\,D_t x = c''.$$

If we multiply each of these equations by the variable which it does not contain, and take their sum, it will reduce to

(9) $$c' x + c'' y + c z = 0.$$

This is the equation of a plane passing through the origin of co-ordinates, or centre of attraction; and hence the co-ordinates of the body M are also the co-ordinates of a plane; and therefore the curve described by M is always found in the same plane, or it is a

plane curve. It will not be necessary, then, to retain z in our discussion of the subject; and by putting it equal to nothing our investigation will be much simplified, and the conclusions none the less general.

4. Putting z and Z equal to nothing in equations (1) and (8), we obtain

(10) $D_t^2 x + X = 0$, (11) $D_t^2 y + Y = 0$, (12) $y\,D_t x - x\,D_t y = c$.

The integral of (12) gives

$$(13) \qquad \int (y\,dx - x\,dy) = ct + C.$$

5. Let us now interpret equation (13). In Fig. 1 let PMY represent the curve described by the body M; S, the centre of attraction; SY and SP, the axes of y and x respectively; $PSM = \varphi =$ the angle which the radius-vector makes with the axis of x, and the other lines as marked in the figure. Then the area $PXM = \int y\,dx'$. But $x' = SP - x$; and since SP is constant, we have $dx' = -\,dx$. Substituting this value in the above equation, we find the area $PXM = -\int y\,dx$. The sector PSM is made up of the area PXM and the triangle SMX. Therefore sector $PSM = SMX + PXM = \frac{1}{2}xy + \int y\,dx' = \frac{1}{2}xy - \int y\,dx$. This being differentiated and reduced, gives

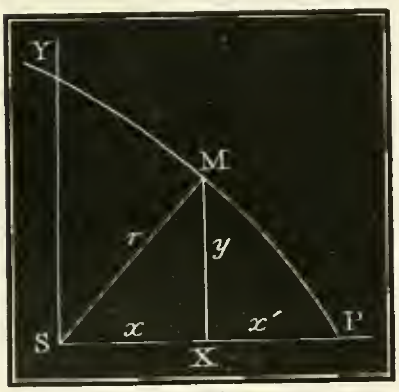


Fig. 1.

$$d\,(\text{sector } PSM) = -\frac{y\,dx - x\,dy}{2} = -\tfrac{1}{2}c\,dt, \text{ from (12)}.$$

We therefore have

$$(14) \qquad 2 \text{ sector } PSM = -\int (y\,dx - x\,dy) = -ct + C.$$

Had we chosen the sector SYM, we should have found it equal to $+\int (y\,dx - x\,dy)$, the only difference being in the sign of the expression. In the first case the area is reckoned from the axis SP, and in the second it is reckoned from the axis SY. We therefore

infer that equation (13) is an expression for the area passed over by the radius-vector. This area ought to be equal to nothing when $t = 0$; hence $C = 0$. This reduces (13) to

$$(15) \qquad \int (y\,dx - x\,dy) = ct.$$

Since c is constant (15) makes known this law; *The area passed over by the radius-vector is proportional to the time.*

6. This was found to hold true in the motion of the planet Mars, by the celebrated JOHN KEPLER; and he extended it by analogy to all the planets. It has since been called KEPLER'S *second law.* This law admits of a generalization wholly unknown to its discoverer; since it is altogether independent of the form of the curve described, and of the nature of the force, either attractive or repulsive. All that is necessary is, that the force be directed along a single line, and that line the one which connects the centres of gravity of the central and moving bodies.

7. To determine the constant c, let $t = 1$ in equation (15), and we find c equal to twice the area passed over in the unit of time.

8. By referring to Fig. 1 we easily find

$$(16) \qquad x = r\cos\varphi, \quad y = r\sin\varphi.$$

Differentiating with respect to the time, t, we obtain

$$(17) \quad D_t x = D_t r\cos\varphi - r\sin\varphi\, D_t\varphi, \quad D_t y = D_t r\sin\varphi + r\cos\varphi\, D_t\varphi.$$

Substituting these values in equation (12) it reduces to

$$(18) \qquad r^2 D_t\varphi = -c.$$

Or changing the axis from which the area is reckoned (Art. 5) we have

$$(19) \qquad r^2 D_t\varphi = c, \qquad\qquad (20) \quad D_t\varphi = \frac{c}{r^2}.$$

Since φ is the space described at the unit of distance, it follows that $D_t\varphi = \omega$, is the velocity at the unit of distance, or it is the angular velocity of the body M. We hence see that (20) makes known this

law, namely, *The angular velocity varies inversely as the square of the distance.* According to NEWTON, the force of gravity in the solar system varies inversely as the square of the distance; hence, in the case of nature, *the force of gravity varies directly as the angular velocity.* But equation (20) comprehends more, for it is as general as the law given by (15), from which it was derived.

9. Equation (19) can be used in computing the time when the angle is given, and *vice versa,* knowing the form of the orbit.

(21) $$t = \frac{1}{c} \int r^2 \, d\varphi, \qquad (22) \quad \varphi = c \int \frac{dt}{r^2}.$$

If we suppose r^{-2} to be a function of t, as it in general is, we can have $\varphi = c \int f(t)\, dt$ (23), and this can be obtained in the form of a series.

10. To obtain a second integral of equations (10) and (11), multiply the first by $2\,dx$, and the second by $2\,dy$, take their sum, and integrate, and we obtain

(24) $$D_t x^2 + D_t y^2 = D_t s^2 = v^2 = C - 2 \int (X\,dx + Y\,dy).$$

Substitute the values of X and Y derived from equations (2), where $\cos\alpha$ and $\cos\beta$ correspond to $\cos\varphi$ and $\sin\varphi$ in equations (16), and $\cos\varphi = \frac{x}{r}$, $\sin\varphi = \frac{y}{r}$, and we obtain

(25) $$D_t x^2 + D_t y^2 = C - 2\int \left(R.\frac{x}{r}.dx + R.\frac{y}{r}.dy\right) = C - 2\int R\left(\frac{x\,dx + y\,dy}{r}\right).$$

But $r^2 = x^2 + y^2$, and $r\,dr = x\,dx + y\,dy$, which being substituted in (25) gives

(26) $$D_t x^2 + D_t y^2 = C - 2\int R\,dr.$$

Substituting for $D_t x$ and $D_t y$ their values given by equations (17), we find

(27) $$r^2 D_t \varphi^2 + D_t r^2 = C - 2\int R\,dr.$$

Equations (18) and (19) both give $r^2 D_t \varphi^2 = \frac{c^2}{r^2}$, and this value substituted in (27) gives

(28) $$\frac{c^2}{r^2} + D_t r^2 = C - 2 \int R\, dr.$$

Differentiating and dividing by $2\, dr$, and transposing, we obtain

(29) $$R = \frac{c^2}{r^3} - D_t^2 r.$$

11. To interpret this equation, let us recur to equation (20) which gives

(30) $$\textit{angular velocity} = D_t \varphi = \omega = \frac{c}{r^2}.$$

That force which is exactly equal and directly opposed to the force, which, placed at S, Fig. 1, would cause M to revolve in a circle at the same distance $r = SM$, with the angular velocity at M, is called the *centrifugal force* at M. When the body revolves in a circle, the centrifugal force is equal to the square of the velocity divided by the radius. Since ω denotes the velocity at the unit of distance, ω will represent the velocity at the distance r, in a circle; and consequently *centrifugal force* $= \frac{\omega^2 r^2}{r} = \omega^2 r$. Substituting for ω its value given by (30), and we obtain

(31) $$\textit{centrifugal force} = f = \frac{c^2}{r^3}.$$

This force which we have represented by f, is that part of the value of R in (29) which regulates the angular velocity of the body; for that part of the force which is represented by $D_t^2 r$ acts in the direction of the radius-vector, and cannot affect ω except it be indirectly through its action in increasing and decreasing the length of the radius-vector. Hence, whatever force R exerts that exceeds or falls short of f is expended in changing the distance of M from the centre of attraction. This is termed the *paracentric force*; and the velocity due to it the *paracentric velocity* ($= D_t r$). When $D_t^2 r$ increases the value of r, it should be affected with the minus sign, and *vice versa*. When $D_t^2 r$ is changing from plus to minus, and con-

versely, it must pass through zero; hence to find that point in a curve where the paracentric force is nothing, take the second differential coefficient of the radius-vector with respect to the time, and put it $= 0$, and the resulting equation will make known the point. As an example take the ellipse whose equation is

$$(32) \qquad r = \frac{k^2}{1 + e \cos\varphi}.$$

$$(33) \quad D_t r = D_\varphi r \, D_t \varphi = D_\varphi r \, . \frac{c}{r^2} = \frac{e\,k^2 \sin\varphi}{(1 + e\cos\varphi)^2} \cdot \frac{c}{r^2} = \frac{e\,c \sin\varphi}{k^2}.$$

$$(34) \; D_t^2 r = D_\varphi (D_t r) \, D_t \varphi = D_\varphi (D_t r) \, . \frac{c}{r^2} = \frac{e\,c^2 \cos\varphi}{k^2 r^2} = 0, \text{ or } \varphi = 90°.$$

And (33) gives for the paracentric velocity at the point, $\frac{e\,c}{k^2}$.

12. Equation (29) gives, since $D_t^2 r = D_\varphi{}^2 r \, D_t \varphi^2 = \frac{c^2}{r^4} \, . \, D_\varphi{}^2 r$,

$$(35) \qquad R = \frac{c^2}{r^4} (r - D_\varphi{}^2 r).$$

If we let $r = u^{-1}$ (36), we have $D_t r = D_\varphi r \, D_t \varphi = \frac{c\,D_\varphi r}{r^2} = - c\,D_\varphi u$.

This value of $D_t r$ substituted in (28) gives, since $d r = - \frac{d u}{u^2}$,

$$(37) \qquad c^2 (u^2 + D_\varphi u^2) = C + 2 \int \frac{R\,d u}{u^2}.$$

Differentiating and dividing by $2\,d u$, we find

$$(38) \qquad R = c^2 u^2 (u + D_\varphi{}^2 u).$$

Equations (35) and (38) are useful for finding the law of the force when the equation of the curve is given. As an example, let the given curve be a common parabola, whose equation is

$$(39) \quad \frac{1}{r} = u = \frac{1 + \cos\varphi}{2\,q}, \quad D_\varphi{}^2 u = - \frac{\cos\varphi}{2\,q} \text{ and } u + D_\varphi{}^2 u = \frac{1}{2\,q}.$$

Substituting in (38) we obtain

$$(40) \qquad R = \frac{c^2 u^2}{2\,q} = \frac{c^2}{2\,q\,r^2}.$$

Hence the force R varies inversely as the square of the distance; a problem which NEWTON first solved.

13. To solve the inverse problem, namely, *having given the law of the force to determine the form of the orbit described,* (the solution of which NEWTON has not transmitted to us,) we shall proceed as follows: — Suppose R to vary according to NEWTON's hypothesis, viz.: directly as the mass and inversely as the square of the distance; we then have $R = \frac{m}{r^2}$, (41). Let $r = 1$, and $R = m =$ the force at the unit of distance. This value of R substituted in (26) and (28) gives

(42) $$D_t x^2 + D_t y^2 = D_t s^2 = v^2 = C + \frac{2m}{r},$$

(43) $$\frac{c^2}{r^2} + D_t r^2 = C + \frac{2m}{r}.$$

But $D_t r = D_\varphi r \, D_t \varphi = D_\varphi r \, . \frac{c}{r^2}$, and hence, by putting $C = h$

(44) $$\frac{c^2}{r^4}(r^2 + D_\varphi r^2) = h + \frac{2m}{r}.$$

Reducing this equation, and we find

(45) $$c \, D_\varphi r = r\sqrt{h r^2 + 2 m r - c^2}.$$

To facilitate the integration, put

(46) $$r = \frac{c}{p + \frac{m}{c}}.$$

This reduces (45) to

(47) $$c \, D_\varphi p = -\sqrt{h c^2 + m^2 - c^2 p^2}.$$

Inverting and integrating,

(48) $$\varphi + \theta = \int \frac{-c \, dp}{\sqrt{h c^2 + m^2 - c^2 p^2}} = \cos^{-1} \frac{c p}{\sqrt{h c^2 + m^2}},$$

θ being a constant. This equation gives

(49) $$c p = \frac{c^2}{r} - m = \sqrt{h c^2 + m^2} \cos(\varphi + \theta).$$

14. In order to determine what curve is described, we shall put (49) under the form

$$(50)\quad r = \frac{c^2}{m + \sqrt{h\,c^2 + m^2}\cos(\varphi + \theta)} = \frac{\frac{c^2}{m}}{1 + \sqrt{\frac{h\,c^2}{m^2} + 1}\cos(\varphi + \theta)}.$$

Equation (50) is of the form of the general equation of the conic sections, referred to the focus as origin, from which we infer that the centre of attraction is at the focus, and it gives

$$(51)\quad \frac{c^2}{m} = p = \text{semi-parameter.} \quad \sqrt{\frac{h\,c^2}{m^2} + 1} = e = \text{eccentricity.}$$

The factor $\frac{c^2}{m^2}$, being a square, is necessarily positive in the expression $\frac{h\,c^2}{m^2}$; and hence the sign of $\frac{h\,c^2}{m^2}$ will depend upon the sign of h. When h is positive, e will be greater than unity, and the curve described will be a hyperbola; the angle $\varphi + \theta$ being reckoned from the vertex. Had we given the radical in (45) the minus sign, the curve would still be a conic section, but $\varphi + \theta$ would be counted from a different point. When h is negative e will be less than unity, and the curve will be an ellipse; and when $h = 0$, $e = 1$, and the curve is a parabola. When h is negative and equal to $\frac{m^2}{c^2}$, $e = 0$, and the curve is a circle; when h is positive and equal to $\frac{m^2}{c^2}$, $e = \sqrt{2}$, and the curve is an equilateral hyperbola.

15. Let us resume the equation (51), and squaring

$$e^2 = \frac{h\,c^2}{m^2} + 1 = \frac{b^2}{a^2} + 1,$$

for the hyperbola; and $\frac{b}{a} = \frac{c}{m}\sqrt{h}$ (52). We also have $p = \frac{c^2}{m} = -a(1 - e^2)$, and multiplying by a we have

$$\frac{a\,c^2}{m} = -a^2(1 - e^2) = a^2 e^2 - a^2 = b^2,\ \frac{b}{a} = \frac{c^2}{b\,m} = \frac{c}{m}\sqrt{h},$$

by (52) $\therefore\ b = \frac{c}{\sqrt{h}}$ (53), and $a = \frac{m}{h}$ (54).

When h is negative, we have

$$e^2 = 1 - \frac{h\,c^2}{m^2} = 1 - \frac{b^2}{a^2}, \quad \frac{b}{a} = \frac{c}{m}\sqrt{h},$$

and we consequently have the same expression for the semi-axes as (53) and (54). Since m is constant, the values of the semi-axes will depend upon those of h and c.

16. Let us now determine the value of the constant h. By comparing equation (42) we see that $h = C$,

$$(\text{velocity})^2 = v^2 = h + \frac{2\,m}{r}.$$

When $r = r_1$, let $v = V$, and we have

$$V^2 = h + \frac{2\,m}{r_1}, \qquad h = V^2 - \frac{2\,m}{r_1}. \tag{55}$$

When the curve is a parabola we have seen that $h = 0$, and hence $V^2 = \frac{2\,m}{r_1}$ (56). This equation gives the velocity in a parabola at any distance r_1 from the centre of attraction, or focus of the curve. When h is positive, we have $V^2 - \frac{2\,m}{r_1}$ positive, and $V^2 > \frac{2\,m}{r_1}$. That is, *the velocity in a hyperbola, at the same distance from the centre of attraction, is greater than in the parabola.*

When h is negative, $V^2 - \frac{2\,m}{r_1}$ is negative, and $V^2 < \frac{2\,m}{r_1}$. That is, *the velocity in an ellipse, at the same distance from the centre of attraction, is less than in a parabola.*

17. Equation (50) gives the demonstration of KEPLER'S *first law*, viz.: that the planets move in ellipses about the sun, which is situated in one of their foci. Article 16 shows that, had their initial velocities been greater, they would have described either parabolas or hyperbolas.

Let S, Fig. 2, be the centre of attraction, $A\,M$ the curve described, and $S\,A$ equal to r. Take the angle $A\,S\,M = d\,\varphi =$ the angle

described in the time dt, and then $AM = ds =$ the space described in the same time. The arc $AB = r\,d\varphi$. But the triangle ABM may be regarded as rectilinear, and right-angled at B; and if we put the angle $AMB = \varepsilon$, we shall have $AB = AM \sin AMB$; or $r\,d\varphi = ds \sin \varepsilon$. Multiplying by r, we have $r^2 d\varphi = c\,dt = r\,ds \sin \varepsilon$; or $c = r\,D_t s \sin \varepsilon = rv \sin \varepsilon$ (57). When $r = r_1$ and $v = V$,

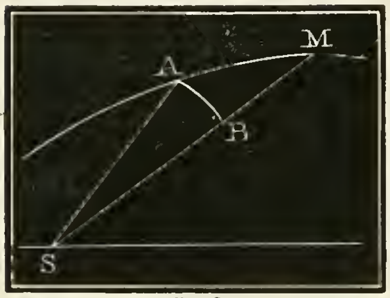


Fig. 2.

let $\varepsilon = \varepsilon_1$; we shall have $c = r_1 V \sin \varepsilon_1$ (58). That is, when the initial velocity and radius vector are the same, the area described in the unit of time is dependent on the angle which the direction of the initial velocity makes with the radius vector; and since the eccentricity is dependent upon the value of c, (51) it follows, that when r_1, V, and consequently h, are the same, that the eccentricity is dependent upon the same angle, because m is constant for the same body. We also see (53) that the semi-conjugate axis is dependent upon the same angle. But the semi-transverse axis (54) is not dependent on c.

19. Since c is twice the area described in the unit of time, in the ellipse it is evidently equal to twice the area of the ellipse divided by the time of revolution in the ellipse, which call T, and we shall have therefore,

$$c = \frac{2\pi ab}{T}, \quad c^2 = \frac{4\pi^2 a^2 b^2}{T^2} = \frac{4\pi^2 a^4 (1 - e^2)}{T^2}.$$

But from (51) we have semi-parameter $= \frac{c^2}{m} = a(1 - e^2)$, and $m = \frac{c^2}{a(1 - e^2)}$. $\therefore\ m = \frac{4\pi^2 a^3}{T^2}$ (59). Since m is constant for the same body, we see that the time of revolution depends wholly upon the value of the transverse axis; and if a circle and any number of ellipses be described on the same transverse axis, the time of revolution in all of them will be the same.

20. We have seen, Art 14, that when $h = \frac{m^2}{c^2}$, the curve is a circle. Calling r_1 the radius, we have

$$c^2 = \frac{4\,\pi^2 r_1^4}{T^2}, \qquad \frac{m}{c^2} = \frac{4\,\pi^2 r_1^3}{T^2} \times \frac{T^2}{4\,\pi^2 r_1^4} = \frac{1}{r_1}, \quad \text{and}$$

(60) $$\frac{m^2}{c^2} = h = \frac{2\,m}{r_1} - V^2 = \frac{m}{r_1}, \qquad \text{or } V^2 = \frac{m}{r_1},$$

for the velocity in a circle. Calling V_c the velocity in a circle, and V_p the velocity in a parabola, we shall have, from (56) and (60),

(61) $$V_p^2 : V_c^2 :: \frac{2\,m}{r_1} : \frac{m}{r_1} :: 2 : 1. \quad \therefore\ V_p = V_c \sqrt{2}.$$

That is, *The velocity in a parabola, at any point, is equal to the velocity in a circle passing through the same point, into the square root of* 2, *the centre of the circle being at the focus of the parabola.* We can, therefore, find the velocity in a parabola at any point, by knowing its distance from the sun.

21. We have represented the attractive force by $R = \frac{m}{r^2}$. To interpret this expression, let M equal the attraction of the central body, at the unit of distance, and then, according to NEWTON's law, $\frac{M}{r^2}$ will equal the attraction at the distance r. Let m_1 equal the attraction of the revolving body at the unit of distance, then $\frac{m_1}{r^2}$ will equal the attraction at the same distance r. But both these forces tend to draw the bodies towards each other; and hence, the whole attractive force which causes the bodies to approach each other is

(62) $$\frac{M}{r^2} + \frac{m_1}{r^2} = \frac{M + m_1}{r^2} = R.$$

This is the force which we have represented by $\frac{m}{r^2}$. Consequently, $M + m_1 = m$ (63). For any other revolving body we have $M + m_2 = m'$.

22. We have already found $m = \frac{4\,\pi^2\,a^3}{T^2}$. For any other body revolving at the mean distance a', we have $m' = \frac{4\,\pi^2\,a'^3}{T'^2}$. Combining these two equations, we have the proportion

$$(64)\qquad m\,T^2 : m'\,T'^2 :: a^3 : a'^3.$$

When m and m' are nearly equal, as is the case in the solar system, we may regard them equal, and we shall have

$$(65)\qquad T^2 : T'^2 :: a^3 : a'^3.$$

That is, *the squares of the periodic times are to each other as the cubes of the mean distances from the sun.* This is KEPLER'S celebrated third law. From (64) we see that it is not strictly true. We have

$$\frac{a}{a'} = \alpha = \left(\frac{T}{T'}\right)^{\frac{2}{3}} \left(\frac{m}{m'}\right)^{\frac{1}{3}} = \left(\frac{T}{T'}\right)^{\frac{2}{3}} \left(\frac{M + m_1}{M + m_2}\right)^{\frac{1}{3}}.$$

Calling $M = 1$, and $\left(\frac{M + m_1}{M + m_2}\right)^{\frac{1}{3}} = \left(\frac{1 + m_1}{1 + m_2}\right)^{\frac{1}{3}} = 1 + \frac{m_1 - m_2}{3}$, by developing and retaining only the first powers of m_1 and m_2, the ratio of the mean distances, or

$$(66)\qquad \alpha = \left(\frac{T}{T'}\right)^{\frac{2}{3}} \left(1 + \frac{m_1 - m_2}{3}\right).$$

23. When the times of revolution and the mean distances are given, we can find the difference between the masses of the two planets;

for $\qquad M + m_1 = \frac{4\,\pi^2\,a^3}{T^2}, \quad M + m_2 = \frac{4\,\pi^2\,a'^3}{T'^2}, \quad$ and

$$(67)\qquad m_1 - m_2 = 4\,\pi^2 \left(\frac{a^3}{T^2} - \frac{a'^3}{T'^2}\right),$$

where m_1 and m_2 are the masses according to NEWTON'S law.

24. Equations (59) can be employed to find the mass of a planet which has a satellite. Call t and a_1 the time of revolution and mean distance of the satellite, m_s its mass, and m_p the mass of the planet. We obtain

(68) $$M + m_p = \frac{4\,\pi^2\,a^3}{T^2}, \quad m_p + m_s = \frac{4\,\pi^2\,a_1^{\,3}}{t^2}\text{; and hence}$$

(69) $$M + m_p : m_p + m_s :: \frac{a^3}{T^2} : \frac{a_1^{\,3}}{t^2} :: \left(\frac{a}{a_1}\right)^3 : \left(\frac{T}{t}\right)^2.$$

By subtraction (68) gives

(70) $$M - m_s = 4\,\pi^2\left(\frac{a^3}{T^2} - \frac{a_1^{\,3}}{t^2}\right)\text{; or } m_s = M - 4\,\pi^2\left(\frac{a^3}{T^2} - \frac{a_1^{\,3}}{t^2}\right).$$

All the quantities a, a_1, T, t, being derived from observation, m_s is known, which substituted in (69) will give the value of m_p.

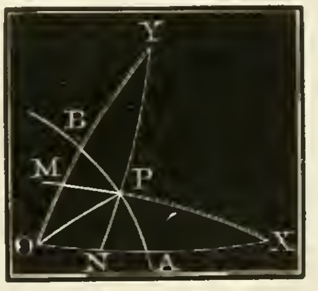
ON SPHERICAL ANALYSIS.*

By George Eastwood, Saxonville, Mass.

Proposition I.

To find the equation of a great circle of the sphere, which passes through the origin, and which makes a given angle with the axis of x.

Let $O\,X$, $O\,Y$, be quadrantal arcs of great circles of the sphere intersecting each other in the point O; and let $O\,P$ be the great circle whose equation is required. Through any point P, pass the great circle arcs $X\,P\,M$, $Y\,P\,N$, and designate $O\,N$ by x, $O\,M$ by y, $O\,P$ by r, the angle $Y\,O\,X$ by ω, and the angle $P\,O\,X$ by φ. Then, in the spherical triangle $X\,O\,M$, we have



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* The following are among the works and memoirs I have had occasion to consult. — *Grundriss der Sphärick Analytischen*, by Professor Gudermann, of Cleve, published about 1830, in which, so far as I know, the idea of expressing the equations of spherical curves in tangent-functions of their rectangular co-ordinate arcs, was first conceived and developed into a system. — Two elaborate memoirs *On the Equations of Loci traced on the Surface of the Sphere*, by the late Professor Davies, of the Royal Military Acade-

$$\cos XM = \cos OM \cos OX + \sin OM \sin OX \cos \angle O,$$
$$= \sin y \cos \omega, \text{ since } OX = \text{a quadrant},$$
$$\cos OM = \cos y = \cos XM \cos XO + \sin XM \sin XO \cos \angle X,$$
$$= \sin XM \cos X,$$
$$= (1 - \sin^2 y \cos^2 \omega)^{\frac{1}{2}} \cos X,$$
$$\cos^2 y = (1 - \sin^2 y + \sin^2 y \sin^2 \omega) \frac{1}{1 + \tan^2 X},$$
$$\tan^2 X = \sin^2 \omega \tan^2 y,$$

(1) $$\tan X = \sin \omega \tan y.$$

For similar reasons, the triangle $Y O N$ gives

(2) $$\tan Y = \sin \omega \tan x;$$

(3) $$\therefore \frac{\tan y}{\tan x} = \frac{\tan X}{\tan Y}.$$

But the triangle $O P X$ gives

$$\cos PX = \sin r \cos \varphi,$$
$$\cos r = \sin PX \cos X,$$
$$= (1 - \sin^2 r \cos^2 \varphi)^{\frac{1}{2}} \cos X,$$
$$\therefore \frac{1 + \tan^2 X}{1 + \tan^2 r} = 1 - \frac{\tan^2 r}{1 + \tan^2 r} + \frac{\tan^2 r \sin^2 \varphi}{1 + \tan^2 r}.$$

(4) $$\therefore \tan X = \sin \varphi \tan r.$$

Likewise, the triangle $O P Y$ gives

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my, Woolwich, England, published in 1832 in the XIIth Volume of the Transactions of the Royal Society of Edinburgh. — "Researches on Spherical Geometry," by the late Professor GILL, published in 1836, in the *Mathematical Miscellany*. — Two Geometrical Memoirs *On the General Properties of the Cone of the Second Degree*, and *on Spherical Conics*, by CHASLES; translated by Rev. CHARLES GRAVES, Fellow and Tutor of Trinity College, Dublin, with an Appendix *On the Application of Analysis to Spherical Geometry*, and published in 1841. — *Essai de Géomètrie Analytique de la Sphère*, by M. BORGNET, published by BACHELIER in 1847. I only know of this work through references to it, all my efforts to procure a copy having been unsuccessful. — Memoir *On Spherical Analysis*, by M. VANNSON, published in TERQUEM and GERONO's *Nouvelles Annales de Mathématiques*, Tome XVII.

$$\tan Y = \sin(\omega - \varphi)\tan r. \tag{5}$$

Hence, by virtue of (3),

$$\tan y = \frac{\sin\varphi}{\sin(\omega - \varphi)}\tan x, \tag{6}$$

is the required equation, which holds for any axes.

If the axes of reference be rectangular, then $\omega = \frac{\pi}{2}$, and (6) reduces to

$$\tan y = \tan\varphi \tan x. \tag{7}$$

Cor. 1. In a plane, the equation of a straight line passing through the origin, when referred to oblique axes, is

$$y = \frac{\sin\varphi}{\sin(\omega - \varphi)}x, \tag{8}$$

and when referred to rectangular axes it is

$$y = \tan\varphi \,.\, x; \tag{9}$$

therefore (6) and (7) are remarkably elegant types of (8) and (9).

Cor. 2. In the triangles ONY, OMX, we have

$$\tan YNX = \frac{\tan\omega}{\cos x},$$

$$\tan YMX = \frac{\tan\omega}{\cos y},$$

$$\therefore \frac{\cos y}{\cos x} = \frac{\tan N}{\tan M}.$$

Assuming the second member to be constant, this equation will define the position of the point P.

Remark. — In geography and astronomy, when we wish to fix the position of a point upon the earth or in the heavens, we usually refer it to the equator and the first meridian, as axes of reference; and then determine the point from its latitude and longitude, or its declination and right ascension. But, in place of the arc of latitude or of declination, we may also take for ordinate the projection of

this arc upon the first meridian. The latter method of determining the position of a point on the sphere has the advantage of giving symmetrical results with respect to x and y, and of rendering it more easy to compare them with the analogous ones in a plane. In order that we may readily distinguish these two systems of co-ordinates, the one from the other, we propose to call the first system geographical, and the second geometrical co-ordinates. The formula for passing from one system to the other is readily found by the resolution of a right spherical triangle. For, if we call $\bar{y}$ the latitude, and y its projection upon the first meridian, or axis of y, then

$$\tan \bar{y} = \tan y \cos x, \tag{5}$$

in which x denotes the longitude of the proposed point.

There is a particular case where the geometrical co-ordinates will not determine the point. This case occurs when $x = \frac{\pi}{2}$, for then y is also equal to $\frac{\pi}{2}$; and the two arcs perpendicular to the axes, which give, in general, the position of the point at their intersection, confound themselves in this specific case. Under these circumstances it will be necessary to take the latitude of the point to fix its position.

[To be Continued.]

THE LAWS OF FALLING BODIES, ADAPTED TO ELEMENTARY INSTRUCTION.

By Thomas Sherwin, Principal of the English High School, Boston.*

I. When a body moves over equal spaces in equal times, it is said to move *uniformly*, or with *uniform velocity*.

* The writer claims no originality in this article. It is merely a compilation, with some modifications and developments, designed to give the learner a simple yet demonstrative exposition of the principles of uniformly accelerated motion, particularly with respect to falling bodies.

This is the kind of motion which results from a force that acts only for an instant, as a momentary impulse, after which the body is abandoned to itself. Under such circumstances, the body must move uniformly and in a straight line. For, in consequence of inertia, the body, unless acted upon by some new force, can neither increase nor diminish its velocity, nor can it change the direction of its motion.

The *velocity* of a body is the space which that body describes uniformly in a unit of time, as a second.

Let u be the velocity, or the space which the body describes in a second; then $t u$ is evidently the space described in t seconds. Calling this space s, we have

(1) $s = ut$; from which we derive

(2) $t = \frac{s}{u}$, and (3) $u = \frac{s}{t}$.

Hence, in uniform motion,

The space is equal to the product of the time and velocity; the time is equal to the quotient of the space divided by the velocity; and the velocity is equal to the quotient of the space divided by the time.

It is also evident that:

When the velocities of two bodies are the same, the spaces are as the times; and, when the times are the same, the spaces are as the velocities.

II. A body continuously acted upon by a constant force moves with a *uniformly accelerated velocity.* For, a body free to move, and urged by such a force, retains all the velocity that has at any time already been imparted to it, and receives equal increments of velocity in equal increments of time.

Although the force of gravity varies inversely as the square of the distance from the earth's centre, yet, within moderate distances from the earth's surface, this force may be regarded as uniform.

Hence, a body near the surface of the earth, and falling under the influence of gravity, descends with uniformly accelerated velocity.

In the descent of a heavy body, the *accelerating force* is represented by the velocity generated in a unit of time. Taking a second as the unit of time, its measure is the velocity per second generated in a second.

Let t be the time of descent estimated in seconds, and k the same time, estimated in instants so small that, during any one of them, the motion may be considered uniform. Suppose that gravity communicates to the body at the commencement of each instant all the velocity it can impart during that instant. Then, if n is the number of these instants in a second,

(1) $$k = nt.$$

Let i be the space which the accelerating force can make the body pass over in an instant; then, i, $2i$, $3i \ldots\ldots ni \ldots\ldots ki$, will be the spaces passed over during the successive instants of the time t.

Suppose that, at the end of the first second, the accelerating force ceases to act; the body, which passed over the space ni, in the nth instant, would then move uniformly, and in the succeeding second would pass over the space $n^2 i$. If, therefore, we designate by g the accelerating force, we have

(2) $$g = n^2 i.$$

If at the end of the time t we suppose the action of the accelerating force again to cease, the body which passed over the space ki in the kth instant, would again move uniformly, and, in the succeeding second, would pass over the space nki, and this space measures the velocity acquired at the end of the time t. Calling this velocity v, we have $v = nki$. If in this value of v we put nt instead of k, and in the result, g instead of $n^2 i$, we have

(3) $$v = gt.$$

Hence, *The velocity acquired at the end of any time is equal to the accelerating force multiplied by the time in seconds.*

The whole space passed over is the sum of the arithmetical progression, i, $2\,i$, $3\,i$, &c. given above. Calling this space s,

$$s = (i + k\,i)\,\frac{k}{2} = (1 + k)\,\frac{k\,i}{2}.$$

The instants into which the time t was conceived to be divided were supposed to be so small that, during any one of them, the motion might be regarded as uniform. But as gravity acts continuously, the time during which the velocity can be regarded as uniform must be infinitely small, and therefore the number of instants in the time t must be infinitely great.

Hence k is infinitely great with respect to 1; this latter may, therefore, be omitted, and

(4) $$s = \frac{k^2\,i}{2} = \frac{n^2\,t^2\,i}{2} = \frac{g\,t^2}{2}.$$ Hence,

The space is equal to half the accelerating force multiplied by the square of the time in seconds.

Let v and v' be the respective velocities acquired in the times t and t'. Then

$$v = g\,t, \text{ and } v' = g\,t';$$

$$\therefore\ v : v' = g\,t : g\,t' = t : t'.$$ That is,

The final velocities are to each other as the times.

Let s and s' be the spaces passed over in the times t and t'; then

$$s = \frac{g\,t^2}{2}, \text{ and } s' = \frac{g\,t'^2}{2};$$

$$\therefore\ s : s' = \frac{g}{2}\,t^2 : \frac{g}{2}\,t'^2 = t^2 : t'^2 = v^2 : v'^2.$$ That is,

The entire spaces are to each other as the squares of the times, or as the squares of the final velocities.

The space passed over in t seconds being $\frac{g}{2}\,t^2$, and that passed over in $t - 1$ seconds being $\frac{g}{2}\,(t - 1)^2$, their difference, $\frac{g}{2}\,(2\,t - 1)$, desig-

nates the space passed over in any one second. Substituting for t successively, 1, 2, 3, 4, &c., and designating the results by s_1, s_2, s_3, s_4, &c., we have,

$$s_1 = \frac{g}{2} \times 1,\ s_2 = \frac{g}{2} \times 3,\ s_3 = \frac{g}{2} \times 5,\ s_4 = \frac{g}{2} \times 7, \text{ \&c.} \quad \text{Hence,}$$

$$s_1 : s_2 : s_3 : s_4, \text{ \&c.} = 1 : 3 : 5 : 7, \text{ \&c.} \quad \text{That is,}$$

The spaces passed over in the separate successive seconds are as the successive odd numbers, 1, 3, 5, 7, &c.

The equation $s_1 = \frac{1}{2} g$, given above, designates the space passed over in the first second of descent, and gives $2 s_1 = g$. But g is the measure of the accelerating force, or the velocity acquired at the end of the first second.

Hence, *The velocity at the end of the first second is twice the space passed over during that second.*

By experiments with the pendulum at Paris, g has been found equal to 32.18 feet. Other experiments have made it equal to $32\frac{1}{6}$ feet, which may be considered as sufficiently accurate for common use, although the value will vary slightly for different latitudes.

The final velocity, as we have seen, is $g t$. But the whole space passed over is $\frac{1}{2} g t^2$, and the average velocity per second is $\frac{1}{2} g t^2$ divided by t, or $\frac{1}{2} g t$. Hence,

The final velocity per second is always double the average velocity per second for the whole descent.

III. Bringing together the important formulæ, we have,

(1) $v : v' = t : t'$; (2) $s : s' = t^2 : t'^2$;

(3) $s : s' = v^2 : v'^2$; (4) $s_1 : s_2 : s_3 : s_4$, &c. $= 1 : 3 : 5 : 7$, &c.

(5) $v = g t$; $\therefore$ (6) $t = \frac{v}{g}$;

(7) $s = \frac{1}{2} g t^2$; $\therefore$ (8) $t = \sqrt{\frac{2 s}{g}}$.

And by eliminating t from (5) and (7),

(9) $s = \frac{v^2}{2 g}$; $\therefore$ (10) $v = \sqrt{2 g s}$.

IV. If we suppose that the body has the velocity u before being subjected to the accelerating force, we have the following formulæ:

(1) $v = u + g t$; $\therefore$ (2) $t = \frac{v - u}{g}$;

(3) $s = u t + \frac{1}{2} g t^2$; $\therefore$ (4) $t = \frac{-u \pm \sqrt{u^2 + 2 g s}}{g}$;

(5) $s = \frac{v^2 - u^2}{2 g}$; $\therefore$ (6) $v = \sqrt{u^2 + 2 g s}$.

V. If, however, the body is projected vertically upward with the velocity u, in which case v and u are of opposite signs, the formulæ become

(1) $v = u - g t$; $\therefore$ (2) $t = \frac{u - v}{g}$;

(3) $s = u t - \frac{1}{2} g t^2$; $\therefore$ (4) $t = \frac{u \pm \sqrt{u^2 - 2 g s}}{g}$;

(5) $s = \frac{u^2 - v^2}{2 g}$; $\therefore$ (6) $v = \sqrt{u^2 - 2 g s}$.

The time at the end of which the body will cease to rise is found by making $v = 0$ in equation (1), which gives $u = g t \therefore t = \frac{u}{g}$; and the height to which the body will rise is found by making $v = 0$ in equation (5), which gives $s = \frac{u^2}{2 g}$. The value of u just found is the same as that of v in section III., and therefore the values of t and s are equal to those given in the same section. Hence,

If a body be projected vertically upward, it will rise to that height from which it would have to fall in order to acquire the velocity of projection, and the times of ascent and descent will be the same.

When the value of t renders that of v or of s negative, the result indicates that at the end of this time the body is descending again, or that it has descended below the point of projection a distance equal to this space.

THE ELEMENTS OF QUATERNIONS.

[Continued from Page 133.]

V. The Distributive and Associative Principles.

30. Let different operations of the same sort be represented by φ, φ', &c. If $(\varphi + \varphi')\,\varphi'' = \varphi\,\varphi'' + \varphi'\,\varphi''$, and $\varphi\,(\varphi' + \varphi'') = \varphi\,\varphi' + \varphi\,\varphi''$, these operations are said to be *distributive*. Also let $\varphi_1 = \varphi\,\varphi'$, and $\varphi_2 = \varphi'\,\varphi''$; then, if $\varphi_1\,\varphi'' = \varphi\,\varphi_2$, these operations are said to be *associative*.*

31. The general conclusion deduced in § 26 applied to the definition in § 8 shows directly, since vectors are special cases of quaternions, that *vectors are distributive.*

32. If the axes of p, q, and r are coplanar, and β the common intersection of the planes of p, q, and r, another line, α, can always be found, such that $r\,\alpha = \beta$. Then, since $(p + q)\,\beta = p\,\beta + q\,\beta$, we have

(38) $(p+q)\,r\,\alpha = p\,r\,\alpha + q\,r\,\alpha = (p\,r + q\,r)\,\alpha$, or $(p+q)\,r = p\,r + q\,r$;

whence, by (8) and (9),

$$\mathrm{K}(p+q)\,r = \mathrm{K}r\,.\,\mathrm{K}(p+q) = \mathrm{K}r\,(\mathrm{K}p + \mathrm{K}q),$$
$$\mathrm{K}(p+q)\,r = Kp\,r + \mathrm{K}q\,r = \mathrm{K}r\,.\,\mathrm{K}p + \mathrm{K}r\,.\,\mathrm{K}q.$$

Equating the last members of these equations, we have in general, since the conjugate of a quaternion is a quaternion,

(38 *a*) $\mathrm{K}r(\mathrm{K}p + \mathrm{K}q) = \mathrm{K}r\,.\,\mathrm{K}p + \mathrm{K}r\,.\,\mathrm{K}q, \quad r(p+q) = rp + rq.$

33. The three quaternions $\mathrm{S}q$, $\mathrm{V}q$, and r, satisfy the condition of coplanarity of axes, since, by § 19, Ax. $\mathrm{S}q$ may be taken in any direction we please. Therefore, (38) and (38 *a*) give for any two quaternions

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* With greater exactness this may be called the associative principle *of multiplication*, and the equation $(\varphi + \varphi') + \varphi'' = \varphi + (\varphi' + \varphi'')$ the expression of the associative principle *of addition*. Similarly the equation $\varphi + \varphi' = \varphi' + \varphi$ may be called the expression of *the commutative principle of addition*. We see directly in § 8 that quaternions satisfy both these latter equations.

(39) $qr = (Sq + Vq)r = Sq.r + Vq.r = Sq.Sr + Sq.Vr + Vq.Sr + Vq.Vr$,

which is equivalent to the two equations

(39′) $Sqr = Sq.Sr + S(Vq.Vr)$, $Vqr = Sq.Vr + Sr.Vq + V(Vq.Vr)$.

34. As $p+q$ is a quaternion it may be substituted for q in (39), then in general for any three quaternions

(40) $$(p+q)r = S(p+q)Sr + S(p+q)Vr + V(p+q)Sr + V(p+q)Vr$$
$$= (Sp+Sq)Sr + (Sp+Sq)Vr + (Vp+Vq)Sr + (Vp+Vq)Vr;$$

whence, as the first three terms of the last member of (40) satisfy the condition of coplanarity of axes, we have, by §§ 32 and 31 and (39),

(41) $$(p+q)r = Sp.Sr + Sq.Sr + Sp.Vr + Sq.Vr + Vp.Sr + Vq.Sr + Vp.Vr + Vq.Vr$$
$$= pr + qr.$$

By the same process used in obtaining (38a) from (38), we obtain

(41a) $$r(p+q) = rp + rq.$$

Equations (41) and (41a) show that in general *quaternions are distributive.* By an easy extension we have for any number of quaternions,

(42) $$(p+p'+p''+\&c.)\,(q+q'+q''+\&c.) = pq + pq' + \&c. + p'q + p'q' + \&c.$$

Equation (42) is merely a more general expression of *the distributive principle.*

35. Let p and q be coplanar, and β the intersection of their common plane with that of r; then another line, α, can always be found in the plane of r, and two other lines, γ and δ, in the common plane of p and q, such that $r\alpha = \beta$, $q\beta = \gamma$, and $p\gamma = \delta$. Denoting the product qr by q', and pq by p', we have $q'\alpha = \gamma$, and $p'\beta = \delta$; whence

$$pq' = \delta \div \alpha = p'r.$$

This equation may also be written

(43) $$p(qr) = (pq)r, \quad \text{or simply } p\,.\,qr = pq\,.\,r.$$

Taking the conjugates of both members of (43) gives

(43*a*) $\quad Kqr \,.\, Kp = Kr \,.\, Kpq = KrKq \,.\, Kp = Kr \,.\, Kq\,Kp$;

but, as these conjugates are quaternions subject only to the condition that the *last two* written shall be coplanar, we see that (43) is true when only q and r, as well as when only p and q, are coplanar.

36. Let α and β be two vectors, perpendicular to each other, then, as a special case of (26), their product, α_1, will be perpendicular to both α and β, and such that we shall have the three symmetric equations,

(44) $\quad U\alpha \,.\, U\beta = U\alpha_1, \quad U\beta \,.\, U\alpha_1 = U\alpha, \quad U\alpha_1 \,.\, U\alpha = U\beta.$

Let γ be any other vector, and let $q = \beta\gamma$. Then, by (31), $U\alpha_1\gamma$ will represent the operation of turning a line from the direction of γ^{-1} to that of α_1, or, what is the same thing, from that of γ to that of $-\alpha_1$. But also, in the same way, Uq and $U\alpha$ represent the respective operations of turning lines from the direction of γ to that of $-\beta$, and from that of $-\beta$ to that of $-\alpha_1$. Therefore,

$$U\alpha \,.\, Uq = U\alpha q = U\alpha_1\gamma;$$

and since evidently $T\alpha q = T\alpha_1\gamma$, it follows that

(45) $$\alpha q = \alpha_1\gamma, \qquad \text{or } \alpha \,.\, \beta\gamma = \alpha\beta \,.\, \gamma.$$

37. Of any three vectors, α, β, and γ, let one, as α, be expressed, by § 22, as the sum of two other vectors, α_1 and α_2, whose directions are respectively parallel and perpendicular to that of β. Then, as a special case of (43), $\alpha_1 \,.\, \beta\gamma = \alpha_1\beta \,.\, \gamma$, and by (45), $\alpha_2 \,.\, \beta\gamma = \alpha_2\beta \,.\, \gamma$; whence, by (41), for any three vectors,

(46) $$\alpha \,.\, \beta\gamma = (\alpha_1 + \alpha_2) \,.\, \beta\gamma = \alpha_1 \,.\, \beta\gamma + \alpha_2 \,.\, \beta\gamma = \alpha_1\beta \,.\, \gamma + \alpha_2\beta \,.\, \gamma = (\alpha_1 + \alpha_2)\beta \,.\, \gamma = \alpha\beta \,.\, \gamma.$$

38. By (42), we develop the following equation:

$$\begin{aligned} p \,.\, qr &= (Sp + Vp) \,.\, (Sq + Vq)(Sr + Vr) \\ (47) \qquad &= Sp \,.\, SqSr + Sp \,.\, SqVr + Sp \,.\, VqSr + Sp \,.\, VqVr \\ &\quad + Vp \,.\, SqSr + Vp \,.\, SqVr + Vp \,.\, VqSr + Vp \,.\, VqVr. \end{aligned}$$

Since, by § 19, the plane of a given scalar may be considered as coinciding with any plane we please in space, all the terms in the last member of (47) except the last, satisfy one or the other of the conditions of § 35; whence, by (43), (46), and (47),

$$
\begin{aligned}
p \,.\, qr &= \mathrm{S}p\,\mathrm{S}q \,.\, \mathrm{S}r + \mathrm{S}p\,\mathrm{S}q \,.\, \mathrm{V}r + \mathrm{S}p\,\mathrm{V}q \,.\, \mathrm{S}r + \mathrm{S}p\,\mathrm{V}q \,.\, \mathrm{V}r \\
&\quad + \mathrm{V}p\,\mathrm{S}q \,.\, \mathrm{S}r + \mathrm{V}p\,\mathrm{S}q \,.\, \mathrm{V}r + \mathrm{V}p\,\mathrm{V}q \,.\, \mathrm{S}r + \mathrm{V}p\,\mathrm{V}q \,.\, \mathrm{V}r \\
&= (\mathrm{S}p + \mathrm{V}p)\,(\mathrm{S}q + \mathrm{V}q) \,.\, (\mathrm{S}r + \mathrm{V}r) = pq \,.\, r.
\end{aligned}
\tag{48}
$$

Equation (48) shows that in general, *quaternions are associative.*

Mathematical Monthly Notices.

A Manual of Spherical and Practical Astronomy, embracing Nautical Astronomy, and the Theory and Use of Fixed and Portable Astronomical Instruments. Amply illustrated by engravings on wood and steel. By Professor WILLIAM CHAUVENET, of the United States Naval Academy. In two royal octavo volumes. Price $7.50. Philadelphia: J. B. Lippincott & Co. London: Trübner & Co.

The Publishers propose to issue these volumes as soon as the number of subscriptions will warrant the undertaking; and in their "Prospectus" they say: —

"There exists at present no work on *Spherical and Practical Astronomy* in the English language, adapted to the wants of the Practical Astronomer, or even of the advanced university student. While there are many elementary treatises designed as text-books in a collegiate course, some of them admirably adapted for this use, there are none which are intended to carry the student beyond the Elements, and to give him that insight into the general theory, and that familiarity with the practical details of the subject, which are indispensable to the working Astronomer.

"PROFESSOR CHAUVENET, who is well known to the scientific world as an exact investigator and clear expounder of mathematical and astronomical subjects, has undertaken to supply this want. His work will not only be the most complete reference-book on this subject that exists in the English language, but will cover the whole ground occupied by the best modern German treatises on both Spherical and Practical Astronomy. The most recent investigations of American as well as European Astronomers will be incorporated in the work. All the most useful problems will be fully illustrated by numerical examples, based upon numbers derived from actual observation, and carried out in the forms which appear to be most approved among experienced computers."

Our readers need no assurance from us of Professor CHAUVENET'S peculiar fitness for the task he has undertaken.

We learn from the "Synopsis of the Table of Contents," that the first part, on "Spherical Astronomy," will contain thirteen Chapters, and the second part, on the "Theory and Use of Astronomical Instruments." ten Chapters, with an Appendix, containing the "Method of Least Squares."

We ask every one at all interested in the subject of these volumes to send their names to the Publishers, and aid in securing their early issue. Chauvenet's Trigonometry is, of itself, an ample guaranty that the proposed volumes will combine the highest scientific and literary excellence.

Tables of Victoria. Computed with Regard to the Perturbations of Jupiter and Saturn. By F. Brünnow, Ph. Dr., Professor of Astronomy in the University of Michigan, and Director of the Observatory at Ann Arbor. Printed by order of the Board of Regents. New York: B. Westermann & Co. London: N. Trübner & Co. 1857. 4to. pp. 77.

These Tables embody the results of the author's labors upon the asteroid Victoria (sometimes called Clio) since its discovery, September 13, 1850; and are based upon the observations made at the oppositions in 1850, 1852, 1853, 1854, 1856, and 1857. The elements derived from the first five of these oppositions, and the numerical expression of the perturbations produced by Jupiter and Saturn, may be found in the *Astronomische Nachrichten*, No. 1077; and the corrections of these elements, as given by the observations of the opposition of 1857, in the *Astronomical Journal*, No. 108. The smallness of these corrections showed that the resulting elements were sufficiently accurate to form the basis of a set of Tables which are now before us. These Tables, by means of 37 different arguments, give the perturbations of M, the mean anomaly, of $\log r$, the logarithm of the radius vector, and of ζ, the perpendicular to the plane of the asteroid's orbit. These are Hansen's co-ordinates, and the general expressions for their perturbations are found in the *Astronomische Nachrichten*, No. 799. Having found these perturbed co-ordinates, we readily pass to the rectangular co-ordinates, referred to the plane of the equator, and the line of the equinoxes. Abbreviated tables are given for finding an approximate place when only five principal terms of the perturbations are used, by means of which an ephemeris sufficiently accurate for all the purposes of observation can be rapidly computed. When it is remembered that an accurate ephemeris is only needed for a month or so before and after the opposition, while a rough one for the purpose of identification is needed for all the rest of the year, the advantage of this feature will be apparent. The Tables are well arranged for use, and printed in fine style. The author has placed astronomers, as well as the science, under renewed obligations by the publication of this timely and valuable work.

Editorial Items.

The following gentlemen have sent us solutions of the Prize Problems in the November Number, Vol. II., of the Monthly: —

James F. Roberson, Senior Class, Indiana University, Bloomington, answered Problems III. and IV.

George B. Hicks, Cleveland, Ohio, answered Problems III., IV., and V.

William Hinchcliffe, Barre Plains, Mass., answered Problems III. and IV.

D. M. Hudson, Paris, Jennings Co., Indiana, answered Problems I. and II.

David Trowbridge, Perry City, Schuyler Co., N. Y., answered Problems III., IV., and the first part of V.

Henry E. Prindle, Ashtabula, Ohio, answered Problems III. and IV.

We would also acknowledge valuable solutions from Asaph Hall, Esq., Assistant at the Observatory of Harvard College, and James Clark, Esq., Wayne, Me.

THE

MATHEMATICAL MONTHLY.

Vol. II. . . . MARCH, 1860. . . . No. VI.

PRIZE PROBLEMS FOR STUDENTS.

I. Find x from the equation,

$$\sqrt[5]{a+x}+\sqrt[5]{a-x}=b,$$

by quadratics.

II. If a circle be described touching the base of a triangle and the sides produced, and a second circle be inscribed in the triangle; prove that the points where the circles touch the base are equidistant from its extremities, and that the distance between the points where they touch either of the sides is equal to the base.

III. Inscribe the maximum rectangle between the conchoid and its directrix. — Communicated by Prof. Daniel Kirkwood.

IV. Given a cask containing a gallons of wine. Through a cock at the bottom of the cask wine flows out at the rate of b gallons per minute, and through a hole at the top water flows in at the same rate. Supposing the water, as fast as it flows in, to mingle perfectly with the wine, how long before the quantities of wine and water in the cask will be equal? and how much wine will be left in the cask at the end of t minutes? — Communicated by Prof. C. A. Young.

V. Two circles being given in a plane, find geometrically the locus of the points from which chords of similar arcs in the two circles will be seen under the same angle, the chords being perpendicular

to the lines of vision drawn through the centres of the given circles. — Communicated by Prof. Wm. Chauvenet.

The solutions of these Problems must be received by the 1st of May, 1860.

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. III., Vol. II.

The first Prize is awarded to John Q. Holliston, Sophomore Class, Hamilton College, Clinton, N. Y.

The second Prize is awarded to F. E. Tower, Senior Class, Amherst College, Amherst, Mass.

The third Prize is awarded to Frank N. Devereux, Boston, Mass.

Prize Solution of Problem I.

By Frank N. Devereux, Boston, Mass.

If two circles touch each other, any straight line passing through the point of contact cuts off similar parts of their circumferences.

The line joining the centres C and C' will pass through the point of contact B. Let A and A' be the points in which the line passing through the point of contact meets the circumferences. Join A and C, A' and C'. The triangles $A\,B\,C$ and $A'\,B\,C'$ thus formed are isosceles and similar, as is easily seen. The angles at the centres, $A\,C\,B$ and $A'\,C'\,B$, are therefore equal, and are measured by similar parts of the circumferences. Hence the proposition is true. This proposition applies whether the circles are tangent externally or internally, a fact not noticed by any of the competitors.

Prize Solution of Problem II.

Find the four roots of the recurring equation

$$x^4 - \tfrac{5}{2}\,x^3 + 2\,x^2 - \tfrac{5}{2}\,x + 1 = 0.$$

Dividing the given equation by x^2 it becomes

$$(1) \qquad (x^2 + \tfrac{1}{x^2}) - \tfrac{5}{2}(x + \tfrac{1}{x}) + 2 = 0.$$

Put $x + \frac{1}{x} = y$; then squaring, $x^2 + \frac{1}{x^2} = y^2 - 2$.

By substitution we get from (1)

$$y^2 - \tfrac{5}{2}y = 0. \qquad \therefore y = 2\tfrac{1}{2} \text{ or } 0.$$

$$\therefore x + \frac{1}{x} = 2\tfrac{1}{2}, \text{ from which } x = 2 \text{ or } \tfrac{1}{2}.$$

$$\therefore x + \frac{1}{x} = 0, \text{ from which } x = \pm\sqrt{-1}.$$

Or thus, the given equation may be written

$$x^2(x^2 - \tfrac{5}{2}x + 1) + x^2 - \tfrac{5}{2}x + 1 = 0;$$

or, $$(x^2 + 1)(x^2 - \tfrac{5}{2}x + 1) = 0.$$

$\therefore x^2 + 1 = 0$; or $x = \pm\sqrt{-1}$. $\therefore x^2 - \frac{5}{2}x + 1 = 0$; or $x = 2$ or $\frac{1}{2}$.

All the competitors gave one or the other of the above solutions.

Prize Solution of Problem III.

By John Q. Holliston, Hamilton College, Clinton, N. Y.

If $2\cos\theta = u + \frac{1}{u}$, prove that $2\cos 2\theta = u^2 + \frac{1}{u^2}$, $2\cos 3\theta = u^3 + \frac{1}{u^3}\ldots\ldots$ $2\cos n\theta = u^n + \frac{1}{u^n}$; and then find the sum of the series, $\cos\theta + \cos 2\theta + \cos 3\theta$ $\ldots\ldots + \cos n\theta$.

By Euler's formula, and the problem,

$$2\cos\theta = e^{\theta\sqrt{-1}} + \frac{1}{e^{\theta\sqrt{-1}}} = u + \frac{1}{u}.$$

$$\therefore u = e^{\theta\sqrt{-1}}, \ \frac{1}{u} = \frac{1}{e^{\theta\sqrt{-1}}}. \qquad \therefore u^n = e^{n\theta\sqrt{-1}}, \ \frac{1}{u^n} = \frac{1}{e^{n\theta\sqrt{-1}}}.$$

Hence, and by Euler's formula,

$$e^{n\theta\sqrt{-1}} + \frac{1}{e^{n\theta\sqrt{-1}}} = u^n + \frac{1}{u^n} = 2\cos n\theta.$$

Denoting by Σ_n the sums relatively to n, we have

$$2\,\Sigma_n \cos n\,\theta = \Sigma_n u^n + \Sigma_n \frac{1}{u^n} = \frac{u^{n+1} - u}{u - 1} + \frac{\dfrac{1}{u^{n+1}} - \dfrac{1}{u}}{\dfrac{1}{u} - 1};$$

since $u + u^2 +$ &c., and $\frac{1}{u} + \frac{1}{u^2} +$ &c., are geometrical series with ratios u and $\frac{1}{u}$.

Remark. — By dividing the terms of the first fraction, and multiplying those of the second, by $u^{\frac{1}{2}}$, we get

$$\Sigma_n \cos n\,\theta = \frac{u^{n+\frac{1}{2}} - u^{\frac{1}{2}} - \dfrac{1}{u^{n+\frac{1}{2}}} + \dfrac{1}{u^{\frac{1}{2}}}}{2\left(u^{\frac{1}{2}} - \dfrac{1}{u^{\frac{1}{2}}}\right)},$$

$$= \frac{u^{n+\frac{1}{2}} - \dfrac{1}{u^{n+\frac{1}{2}}} - \left(u^{\frac{1}{2}} - \dfrac{1}{u^{\frac{1}{2}}}\right)}{2\left(u^{\frac{1}{2}} - \dfrac{1}{u^{\frac{1}{2}}}\right)} = \frac{\sin\,(n + \frac{1}{2})\,\theta - \sin\frac{1}{2}\,\theta}{2\sin\frac{1}{2}\,\theta}.$$

This form of the result was not given by any of the competitors.

Prize Solution of Problem IV.

John Q. Holliston, Hamilton College, Clinton, N. Y.

Having given the Right Ascensions and Declinations of two stars, to find the formula for the distance between them. Also, find what the distance becomes, when for one star A. R. is $8^{\mathrm{h}}\ 12^{\mathrm{m}}\ 38^{\mathrm{s}}.17$, and Dec. $17°\ 23'\ 49''.8$ north, and for the other A. R. is $13^{\mathrm{h}}\ 28^{\mathrm{m}}\ 19^{\mathrm{s}}.92$, and Dec. $21°\ 12'\ 37''.2$ south.

Let δ and δ' be the declinations of the two stars; then $90° - \delta$ and $90° - \delta'$ will be their co-declinations; and since the right ascensions are measured on the equator, their difference will measure the angle at the pole made by the meridians passing through the two stars. Let H denote this angle, and Δ the distance sought. We have then a spherical triangle in which the sides $90° - \delta$ and $90° - \delta'$ include the angle H, and Δ, the third side, may be found from the formula

$$\cos \Delta = \cos(90° - \delta)\cos(90° - \delta') + \sin(90° - \delta)\sin(90° - \delta')\cos H,$$

$$(1) \qquad = \sin\delta \sin\delta' + \cos\delta\cos\delta'\cos H,$$

$$(2) \qquad = \sin\delta\cos\delta'\left(\frac{\sin\delta'}{\cos\delta'} + \cot\delta\cos H\right).$$

Assume

$$(3) \qquad \cot\delta\cos H = \tan\omega = \frac{\sin\omega}{\cos\omega};$$

then, substituting, (2) becomes

$$(4) \qquad \cos\Delta = \frac{\sin\delta\sin(\delta' + \omega)}{\cos\omega}.$$

Formulas (3) and (4) need only tables of logarithmic sines, cosines, &c.; and we find $\Delta = 86° \; 24' \; 12''.2$.

Prize Solution of Problem V.

By Asher B. Evans, Madison University, Hamilton, N. Y.

In a frustum of any pyramid or cone, the area of a section, parallel to the two bases and equidistant from them, is the arithmetical mean of the arithmetical and geometrical means of the areas of the two bases.

The three sections are evidently similar figures; hence their areas will be as the squares of any homologous lines. Let x, $x - y$, and $x + y$, represent any homologous lines in the upper base, middle section, and lower base, respectively. Then their areas may be represented by $m(x - y)^2$, mx^2, $m(x + y)^2$, respectively. The arithmetical mean of the two bases is

$$\frac{m(x - y)^2 + m(x + y)^2}{2} = m(x^2 + y^2);$$

their geometrical mean is

$$\sqrt{m^2(x - y)^2(x + y)^2} = m(x^2 - y^2);$$

and the arithmetical mean of $m(x^2 + y^2)$ and $m(x^2 - y^2)$ is mx^2, as was to be shown.

Simon Newcomb.
W. P. G. Bartlett.
Truman Henry Safford.

NOTES AND QUERIES.

1. What fraction is that, to the numerator of which if 1 be added, its value will be $\frac{1}{3}$; but if 1 be added to the denominator, its value will be $\frac{1}{4}$?

The successive multiples of $\frac{1}{3}$ are $\frac{2}{6}$, $\frac{3}{9}$, $\frac{4}{12}$, $\frac{5}{15}$, $\frac{6}{18}$, &c.; hence, taking unity from the numerator of each of these, we have the series

$$\tfrac{1}{6},\ \tfrac{2}{9},\ \tfrac{3}{12},\ \tfrac{4}{15},\ \tfrac{5}{18},\ \tfrac{6}{21},\ \text{\&c.},$$

and the required fraction must necessarily be one of this series.

Again, the successive multiples of $\frac{1}{4}$ are $\frac{2}{8}$, $\frac{3}{12}$, $\frac{4}{16}$, $\frac{5}{20}$, $\frac{6}{24}$, &c.; whence taking unity from the denominator of each of these, we get

$$\tfrac{2}{7},\ \tfrac{3}{11},\ \tfrac{4}{15},\ \tfrac{5}{19},\ \tfrac{6}{23},\ \tfrac{7}{27},\ \text{\&c.},$$

a series of fractions in which the required fraction must necessarily be found.

Examining these two series, we find that the third fraction in the second series is the same as the fourth fraction in the first series, and therefore, as the sought fraction must be found in both series, we conclude with the utmost certainty that $\frac{4}{15}$ is the fraction fulfilling the conditions of the question.

2. What fraction is that, whose numerator being doubled, and denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$?

Reasoning as in the preceding example, we must first take the successive multiples of $\frac{2}{3}$; divide the numerator of each multiple by 2, and decrease each denominator by 7; then we have the following series, viz.:—

$$\tfrac{3}{2},\ \tfrac{4}{5},\ \tfrac{5}{8},\ \tfrac{6}{11},\ \tfrac{7}{14},\ \tfrac{8}{17},\ \text{\&c.}$$

Again, take the consecutive multiples of $\frac{3}{5}$, decrease the numera-

tor of each by 2, and divide the denominator by 2; then we have the series

$$\tfrac{2}{5},\ \tfrac{4}{5},\ \tfrac{15}{16},\ \tfrac{10}{10},\ \tfrac{26}{25},\ \tfrac{16}{15},\ \&c.$$

The required fraction must exist in both series, in order that the two conditions in the enunciation may be complied with, and as $\frac{4}{5}$ is the only fraction common to both, we conclude that $\frac{4}{5}$ will satisfy the conditions stated in the question. — Prof. W. Rutherford, Woolwich. — *The Northumbrian Mirror*, 1838.

1. *Isochronous Motions.* — The principle of isochronous motions, as exhibited in the small oscillations of the pendulum, in the oscillations of the hair-spring balance, and in the descent of weights on the cycloid, is not usually set forth in the analytical treatment of these examples with sufficient prominence to be distinctly generalized and abstractly comprehended, though the analysis necessarily involves it.

In the following theorem and demonstration this principle is distinctly presented, and from it may be deduced all the cases that have been separately investigated.

Theorem. — If a material point, constrained to move in any given path, tend to approach a given fixed point in the path, urged along by a force proportional to the length of the path between it and the given point; it will pass from a state of rest at any point whatever to the given point always in the same time; or it will pass from the given point with any velocity whatever to a state of rest in the same time.

Demonstration. — If two material points start at rest in two positions on the path, and if we suppose that the two lengths of the path included between the given fixed point and these positions be divided into the same number of very small parts, the ratio of the corresponding parts, and the ratio of their distances from the given fixed point along the path, will be equal to the ratio of the lengths themselves; hence, the forces acting in corresponding parts will be in the ratio of the parts themselves; hence, the sum of all the

actions of the forces through any number of parts in one length will be to the sum of the actions of the forces in the same number of corresponding parts in the other length in the same ratio; hence, the two velocities which the two material points will have in passing through corresponding parts will be in the ratio of the parts themselves, and these corresponding parts will therefore be passed over in the same time, and any number of them in one length will be passed over in the same time as the same number of the corresponding ones in the other length; hence, the whole lengths will be described in the same time; that is, the two material points will arrive at the given fixed point at the same instant, and with velocities proportional to the lengths of the path they have described. If a material point set out from the given fixed point with any velocity, it is obvious that it will come to rest at that point from which it would attain this velocity in moving back to the given fixed point; and that it will have the same velocities in every part as in moving from rest to the given fixed point, though in a contrary direction; hence the time will be the same.

Since a body, descending upon any path under the action of gravity, is impelled along the path by the force $g \sin \tau$, in which g is the direct force of gravity, and τ the angle made by the direction of the curve at any point with the horizon, it follows, that the times of descent will be the same from all parts of the curve to the point for which $\tau = 0$, or the lowest point of the curve, if the arc s, reckoned from this lowest point, be proportional to the force $g \sin \tau$. The equation $s = 4\, R \sin \tau$ is an equation of the cycloid, a curve which is therefore called the *Tautochrone.* In the common pendulum, the force $g \sin \tau$, which impels the ball along its circular path, is nearly proportional to the path itself when the amplitudes of the vibrations are small. The small motions of the pendulum are therefore nearly isochronous. The small motions of

a body vibrating upon the lowest portion of any curve are, for the same reason, nearly isochronous. When the force of the hair-spring of the watch-balance is proportional to the angle by which it is drawn from its position of equilibrium, the motions of the balance are, by the reasoning above, also isochronous. — W.

2. *Problem.** Given $A = \frac{15\,(E - \sin E)}{9\,E + \sin E}$ (1), and $T = \tan^2 \frac{1}{2} E$, to find

$$(2) \qquad A = \frac{T - \frac{6}{5}\,T^2 + \frac{9}{7}\,T^3 - \frac{12}{9}\,T^4 + \frac{15}{11}\,T^5 - \&c.}{1 - \frac{6}{15}\,T + \frac{7}{25}\,T^2 - \frac{8}{35}\,T^3 + \frac{9}{45}\,T^4 - \&c.}.$$

Dividing both terms of the value of A in (1) by $\cos E$ it becomes

$$(3) \qquad A = \frac{15\,(E \sec E - \tan E)}{9\,E \sec E + \tan E}.$$

Now, $\qquad \sec E = \dfrac{1 + \tan^2 \frac{1}{2} E}{1 - \tan^2 \frac{1}{2} E}$, and $\qquad \tan E = \dfrac{2 \tan \frac{1}{2} E}{1 - \tan^2 \frac{1}{2} E}$.

Substituting these values in (3) we get

$$(4) \quad A = \frac{30\,\{\frac{1}{2} E\,(1 + \tan^2 \frac{1}{2} E) - \tan E\}}{18\,\frac{1}{2} E\,(1 + \tan^2 \frac{1}{2} E) + 2 \tan \frac{1}{2} E} = \frac{30\,\{\frac{1}{2} E\,(1 + T) - T^{\frac{1}{2}}\}}{18\,\frac{1}{2} E\,(1 + T) + 2\,T^{\frac{1}{2}}}.$$

But, $\frac{1}{2} E = \tan^{-1} T^{\frac{1}{2}} = T^{\frac{1}{2}} - \frac{1}{3}\,T^{\frac{3}{2}} + \frac{1}{5}\,T^{\frac{5}{2}} - \frac{1}{7}\,T^{\frac{7}{2}} + \frac{1}{9}\,T^{\frac{9}{2}} - \&c.$;
multiplying this value of $\frac{1}{2} E$ by $1 + T$, and subtracting $T^{\frac{1}{2}}$ from the product, the numerator in (4) becomes

$$30\,(\tfrac{2}{3}\,T^{\frac{3}{2}} - \tfrac{2}{15}\,T^{\frac{5}{2}} + \tfrac{2}{35}\,T^{\frac{7}{2}} - \tfrac{2}{63}\,T^{\frac{9}{2}} + \&c.);$$

and by a similar process the denominator becomes

$$20\,(T^{\frac{1}{2}} + \tfrac{3}{5}\,T^{\frac{3}{2}} - \tfrac{3}{25}\,T^{\frac{5}{2}} + \tfrac{9}{175}\,T^{\frac{7}{2}} - \&c.)$$

$$\therefore\ A = \frac{T^{\frac{3}{2}} - \frac{1}{5}\,T^{\frac{5}{2}} + \frac{3}{35}\,T^{\frac{7}{2}} - \frac{1}{21}\,T^{\frac{9}{2}} + \&c.}{T^{\frac{1}{2}} + \frac{3}{5}\,T^{\frac{3}{2}} - \frac{3}{25}\,T^{\frac{5}{2}} + \frac{9}{175}\,T^{\frac{7}{2}} - \&c.}.$$

If now we divide both terms of this value of A by $T^{\frac{1}{2}} + T^{\frac{3}{2}}$, we shall obtain (2) as was to be done. — Richard Cotter, Professor of Mathematics in Baltimore College.

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* From Gauss's *Theoria Motus*. See Davis's translation, page 46.

3. The series (10) page 87, Vol. I., taken *without limitation*, is always equal to unity. For, putting $\frac{y}{l} = \frac{x-1}{x}$, it becomes

$$n\left(\frac{x-1}{x}\right)^{n-1} - \frac{n(n-1)}{2!}\left(\frac{x-2}{x}\right)^{n-1} + \&c. = S;$$

but in the Calculus of Finite Differences we have

$$\Delta^n u^m = (u+n)^m - n(u+n-1)^m + \frac{n(n-1)}{2!}(u+n-2)^m - \&c.;$$

hence,

$$x^{n-1}(1-S) = \Delta^n (x-n)^{n-1} = 0. \qquad \therefore S = 1.$$

The special case of S, on page 87, was put equal to unity only because it expressed the *probability of a certainty*. We now deduce the same result from its algebraic form alone. — B.

4. Solve the differential equation

$$n^2 \varphi(r) + \frac{\varphi'(r)}{r} + \varphi''(r) = 0.$$

— Airy's *Tides and Waves*. Ency. Metrop.

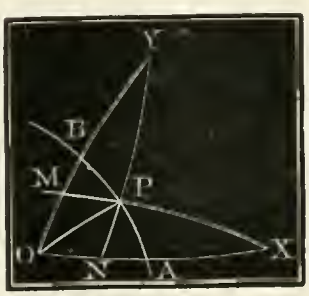
ON SPHERICAL ANALYSIS.

By George Eastwood, Saxonville, Mass.

[Continued from Page 166.]

Proposition II.

Knowing the co-ordinates, O N, O M, *of a point* P, *referred to oblique axes, to find its distance,* O P, *from the origin.*



We have already found, (1) and (4),

$$\tan OP = \tan r = \frac{\tan x}{\sin \varphi} = \frac{\sin \omega \tan y}{\sin \varphi};$$

but equation (3), viz.

$$\frac{\tan y}{\tan x} = \frac{\sin \varphi}{\sin(\omega - \varphi)},$$

gives

$$\tan \varphi = \frac{\sin \omega \tan y}{\tan x + \cos \omega \tan y},$$

$$\sin \varphi = \frac{\sin \omega \tan y}{(\tan^2 x + \tan^2 y + 2 \tan x \tan y \cos \varphi)^{\frac{1}{2}}}.$$

The substitution of this value of $\sin \varphi$ in $\tan r$ gives

(10) $$\tan^2 r = \tan^2 x + \tan^2 y + 2 \tan x \tan y \cos \omega.$$

If the axes be rectangular, then $\omega = \frac{1}{2}\pi$, and (10) reduces to

(11) $$\tan^2 r = \tan^2 x + \tan^2 y.$$

Cor. 1. In a plane we have

(12) $$r^2 = x^2 + y^2 + 2\,x\,y \cos \omega,$$

when the axes are oblique; and

(13) $$r^2 = x^2 + y^2,$$

when the axes are rectangular. Hence, if we agree, in the present and subsequent investigations, to designate the tangent-function of all great circle arcs emanating from the origin, by the symbols of those arcs, we have a perfect type of (12) and (13), in (10) and (11).

Cor. 2. If, with centre O and radius $O\,P =$ constant, we describe a less circle of the sphere, then

(14) $$r^2 = x^2 + y^2 + 2\,x\,y \cos \omega,$$

will be its equation for oblique, and

(15) $$r^2 = x^2 + y^2,$$

for rectangular axes, the origin being at the centre.

Proposition III.

To find the equation of a great circle which intersects the axes at given distances from the origin.

Let A and B be the points in which the great circle cuts the axes of reference. Through any point P in this circle, draw the project-

ing arcs XPM, YPN, and designate OA by α, OB by β, and the other arcs and angles as in Prop. I. Then the triangle, AOB, cut by the spherical transversal MPX, gives the relations

$$\sin BM : \sin BP = \sin \angle P : \sin \angle M,$$
$$\sin OX : \sin OM = \sin \angle M : \sin \angle X,$$
$$\sin AP : \sin AX = \sin \angle X : \sin \angle P.$$

In like manner, the triangle AOB, cut by the spherical transversal NPY, gives

$$\sin AN : \sin AP = \sin \angle P : \sin \angle N,$$
$$\sin OY : \sin ON = \sin \angle N : \sin \angle Y,$$
$$\sin BP : \sin BY = \sin \angle Y : \sin \angle P.$$

Compounding these respective groups of ratios, effacing common factors, and remembering that OX and OY are quadrantal arcs of great circles, we have the further relations

$$\sin BM \,.\, \sin AP = \sin BP \,.\, \sin OM \,.\, \sin AX,$$
$$\sin AN \,.\, \sin BP = \sin AP \,.\, \sin ON \,.\, \sin BY,$$

which, when expressed in symbols, give

$$\frac{\sin(\beta - y) \,.\, \sin AP}{\sin y \,.\, \cos\alpha \,.\, \sin BP} = 1,$$

$$\frac{\sin(\alpha - x) \,.\, \sin BP}{\sin x \,.\, \cos\beta \,.\, \sin AP} = 1.$$

Multiplying these equations together, member by member, and reducing, we obtain

$$\frac{\frac{\sin y}{\cos y}}{\frac{\sin\beta}{\cos\beta}} + \frac{\frac{\sin x}{\cos x}}{\frac{\sin\alpha}{\cos\alpha}} = 1,$$

or,

(16) $$\frac{y}{\beta} + \frac{x}{\alpha} = 1.$$

(17) $$\therefore\, y = -\frac{\beta}{\alpha}x + \beta.$$

As (1) and (2) are independent of ω, either of them may be regarded as the general equation of a great circle of the sphere referred to any axes.

When the axes are rectangular, and when $O P$ is perpendicular to $A B$, then the right triangles $B P O$, $A P O$ give

$$\tan O P = \tan O B \,.\, \cos B O P = \beta \sin \varphi,$$
$$\tan O P = \tan O A \,.\, \cos A O P = \alpha \cos \varphi.$$
$$\therefore \tan \varphi = \frac{\alpha}{\beta} = \frac{1}{\frac{\beta}{\alpha}}.$$

Hence, by (7), the equation of $O P$ is

(18) $$y = \frac{1}{\frac{\beta}{\alpha}}\, x.$$

Cor. 1. In a plane, the equation of a straight line, referred to rectangular axes, is

$$y = t x + b,$$

and the equation of a line perpendicular to this, from the origin, is

$$y = -\frac{1}{t}\, x.$$

If, therefore, we put $\tau = -\frac{\beta}{\alpha}$, (17) and (18) become

(19) $$y = \tau x + \beta,$$

(20) $$y = -\frac{1}{\tau}\, x.$$

Cor. 2. We may give to (19) this form,

(21) $$y = \tau x \pm \varrho\, (1 + \tau^2)^{\frac{1}{2}}.$$

For, if we assume $\varrho = O P$, then

$$\varrho = \beta \cos B O P,$$
$$= \pm \frac{\beta}{(1 + \tau^2)^{\frac{1}{2}}},$$

and

$$\beta = \pm \varrho\, (1 + \tau^2)^{\frac{1}{2}}.$$

Remark.—In the diagram, the great circle is represented as cutting the axis of x *east* of the origin, and hence α is counted positive. If it cut the axis of x *west* of the origin, then α must be counted negative, just as in a plane. Hence the sign of τ, in (19), (20), and (21), must be always counted the reverse of that of α.

NOTES ON ANALYTIC TRIGONOMETRY.

By O. Root, Professor of Mathematics in Hamilton College, Clinton, N. Y.

Let n be any number, and φ any arc whatever with a radius of unity; from the differential calculus we shall have

(1) $$d \cos \varphi = - \sin \varphi \, d \varphi,$$

(2) $$d \cos n \varphi = - n \sin n \varphi \, d \varphi.$$

From these equations we readily get

(3) $$n \frac{d \cos \varphi}{\sin \varphi} = \frac{d \cos n \varphi}{\sin n \varphi}.$$

And since

(4) $\sin \varphi = (1 - \cos^2 \varphi)^{\frac{1}{2}}$, and $\sqrt{(-1)} \sin \varphi = (\cos^2 \varphi - 1)^{\frac{1}{2}}$,

we may write (3) in the following form,

(5) $$n \frac{d \cos \varphi}{(\cos^2 \varphi - 1)^{\frac{1}{2}}} = \frac{d \cos n \varphi}{(\cos^2 n \varphi - 1)^{\frac{1}{2}}}.$$

Integrating (5) we shall get

$$n \log (\cos \varphi + \sqrt{(-1)} \sin \varphi) = \log (\cos n \varphi + \sqrt{(-1)} \sin n \varphi),$$

and this may be written

(6) $$(\cos \varphi + \sqrt{(-1)} \sin \varphi)^n = (\cos n \varphi + \sqrt{(-1)} \sin n \varphi),$$

which is the formula of De Moivre.

Equation (1) gives

$$d \varphi = \frac{- d \cos \varphi}{(1 - \cos^2 \varphi)^{\frac{1}{2}}}.$$

Multiplying both sides by $\sqrt{-1}$, we shall have

(7) $$\sqrt{(-1)}\, d\varphi = \frac{-d\cos\varphi}{(\cos^2\varphi - 1)^{\frac{1}{2}}},$$

the integral of which is

(8) $$\sqrt{(-1)}\, \varphi = \log\,(\cos\varphi + \sqrt{(-1)}\sin\varphi).$$

No constant is required, for when $\varphi = 0$ then $\sin\varphi = 0$, and $\cos\varphi = 1$, whose $\log = 0$. Changing (8) to its exponential form, we get

(9) $$\cos\varphi + \sqrt{(-1)}\sin\varphi = e^{\varphi\sqrt{-1}},$$

which is the formula of Euler.

Equation (6) can be easily derived from (9); if we put $n\varphi$ for φ, we shall get

(10) $$\cos n\varphi + \sqrt{(-1)}\sin n\varphi = e^{n\varphi\sqrt{-1}};$$

then if we raise both sides of (9) to the nth power, we shall have

(11) $$(\cos\varphi + \sqrt{(-1)}\sin\varphi)^n = e^{n\varphi\sqrt{-1}};$$

hence from (10) and (11) we get

$$(\cos\varphi + \sqrt{(-1)}\sin\varphi)^n = \cos n\varphi + \sqrt{(-1)}\sin n\varphi,$$

which is the same as equation (6), or De Moivre's formula deduced from Euler's.

Again, if in equation (9) we put $\varphi = a$, b, and $(a + b)$, successively, we shall get

(12) $$\cos a + \sqrt{(-1)}\sin a = e^{a\sqrt{-1}},$$

(13) $$\cos b + \sqrt{(-1)}\sin b = e^{b\sqrt{-1}},$$

(14) $$\cos(a + b) + \sqrt{(-1)}\sin(a + b) = e^{(a+b)\sqrt{-1}}.$$

Multiplying (12) and (13) together and comparing the product with (14), equating irrational parts with irrational parts, and rational parts with rational parts, we shall get

(15) $$\sin(a + b) = \sin a \cos b + \cos a \sin b,$$

(16) $$\cos(a + b) = \cos a \cos b - \sin a \sin b,$$

which are fundamental equations in trigonometry.

Also, if in (6) we put $n = 2$, we shall get

$$(\cos\varphi + \sqrt{(-1)}\sin\varphi)^2 = \cos 2\varphi + \sqrt{(-1)}\sin 2\varphi;$$

expanding the binomial, and equating irrational parts, we shall get

$$\sin 2\varphi = 2\sin\varphi\cos\varphi,$$

$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi.$$

It is evident that (15) and (16) will give the same results if we put $a = b = \varphi$. Again, if in equation (9) we write $-\varphi$ for φ, we shall have

(17) $$\cos\varphi - \sqrt{(-1)}\sin\varphi = e^{-\varphi\sqrt{-1}};$$

from (9) and (17) we shall obviously get

$$\sin\varphi = \frac{e^{\varphi\sqrt{-1}} - e^{-\varphi\sqrt{-1}}}{2\sqrt{-1}},$$

$$\cos\varphi = \frac{e^{\varphi\sqrt{-1}} + e^{-\varphi\sqrt{-1}}}{2}.$$

If we develop the exponentials in these equations, we shall get

$$\sin\varphi = \varphi - \frac{\varphi^3}{1.2.3} + \frac{\varphi^5}{1.2.3.4.5} - \&c.,$$

$$\cos\varphi = 1 - \frac{\varphi^2}{1.2} + \frac{\varphi^4}{1.2.3.4} - \frac{\varphi^6}{1.2.3.4.5.6} + \&c.$$

Again, if in equation (8) we put $\varphi = 180° = \pi$, we shall get

$$\sqrt{(-1)}\,\pi = \log(-1);$$

therefore

$$\pi = \frac{\log(-1)}{\sqrt{-1}}.$$

For other applications of these equations see Chauvenet's Trigonometry, and Moigno's Differential Calculus.

ON THE INDETERMINATE ANALYSIS.

By Rev. A. D. Wheeler, Brunswick, Maine.

[Continued from Page 57.]

Proposition X. The least value of c for n solutions, in the equation $ax + by = c$, is $c = (n - 1)ab + a + b$.

Demonstration. Let $x = 1$ and $y = 1$; since these are the least values that can be given them. Then will the succeeding values of x be $1 + b$, $1 + 2b$, &c., to $1 + (n - 1)b$, for the nth solution. (Prop. VII.) Substituting for x and y their least values for the nth solution, as thus obtained, we have

$$ax + by = a(1 + (n - 1)b) + b = (n - 1)ab + a + b = c;$$

which is the value required.

Prop. XI. When $c = nab$, the equation $ax + by = c$ admits of only $n - 1$ solutions, in whole numbers.

Dem. Put v for the first value of x. Then will $v + (n - 1)b$ be the nth value as before.

By substitution, $ax + by = av + (n - 1)ab + by = nab$, according to the supposition. Reducing, we obtain $av + by = ab$, which admits of no solution; as shown in Prop. IX. Case 3. All the intermediate values of x will render the equation possible. Therefore the whole number of solutions is $n - 1$.

Prop. XII. The greatest value of c for n solutions is $c = (n + 1)ab$.

Dem. Let $x = v$ be the first value of x, and $x = v + (n - 1)b$ be the nth, as in the preceding proposition. Then

$$ax + by = av + (n - 1)ab + by = (n + 1)ab,$$

according to the supposition. Reducing, we obtain the equation $av + by = 2ab$. This is possible for one solution, as shown in Prop. VIII., and for only one, as shown in Prop. XI. Therefore the equation $ax + by = (n + 1)ab$ is possible for n solutions and no

more. Let $n + 1 = n'$. Now if c were greater than $n'ab$; that is, if $c = n'ab + r$; then, by Prop. VIII., the equation would admit of n' or $n + 1$ solutions. Therefore $(n + 1)\,ab$ is the greatest value of c for n solutions.

Prop. XIII. When $c = nab + ax' + by'$, the equation $ax + by = c$ is always possible for $n + 1$ solutions.

Dem. We have $ax + by = nab + ax' + by'$. Let $x = v$, for its first value. Then for the $(n + 1)$th value we shall have $v + nb$, (Prop. VII.), and the equation becomes

$$av + nab + by = nab + ax' + by', \text{ or}$$
$$av + by = ax' + by';$$

this is clearly possible, since we can always make $v = x'$ and $y = y'$.

Remark. — If we divide $nab + ax' + by'$ by ab, we obtain the quotient n, and the remainder $ax' + by'$. Now the equation $av + by = ax' + by'$, will always admit of one solution, for the reason which has been given. It cannot admit of more than one, since $ax' + by' > ab$; and the least value for two solutions, according to Prop. X., is $c = ab + a + b$. Further, as it has been proved (Prop. XI.) that $c = nab$ will give $n - 1$ solutions; and (Prop. VIII.) that $c > nab$ will give n solutions; we are able to determine, in all cases, the exact number of solutions which any equation of this form admits of, by the following simple

Rule. — Divide c by ab. If there be no remainder, the whole number of solutions will be one less than the quotient. If there be a remainder and that remainder admits of a solution, it will be one more than the quotient. In all other cases it will be equal to the quotient.

Prop. XIV. If the numbers 1, 2, 3, &c., in their order, be substituted for c, until we arrive at the value $c = ab$, the possible equations, of the form $ax + by = ax' + by'$, in this series, will always be equal to the impossible equations, of the form $ax + by = ab - (ax' + by')$.

DEM. Let $a\,x + b\,y = a\,b$,

and $\quad a\,x'' + b\,y'' = a\,b - (a\,x' + b\,y')$, an impossible equation. Subtracting, $a\,(x - x'') + b\,(y - y'') = a\,x' + b\,y'$; which is possible, since we can make $x - x'' = x'$, and $y - y'' = y'$. Thus there is a possible equation for every impossible one of this form, and, of course, the number of each must be equal.

[To be Continued.]

THE ELEMENTS OF QUATERNIONS.

[Continued from Page 175.]

VI. GENERAL FORMULÆ.

39. Two special cases of equations (39′) deserve notice. When q is a tensor or a scalar, they become

(49) $\qquad \mathrm{S}\,q\,r = q\,\mathrm{S}\,r, \qquad$ and $\quad \mathrm{V}\,q\,r = q\,\mathrm{V}\,r.$

And when $r = q$, they become, by (17),

(50) $\qquad \mathrm{S}\,q^2 = (\mathrm{S}\,q)^2 + (\mathrm{V}\,q)^2, \qquad$ and $\mathrm{V}\,q^2 = 2\,\mathrm{S}\,q\,.\,\mathrm{V}\,q.$

The sum and difference of (50) and (35) give

(51) $\qquad (\mathrm{S} + \mathrm{T})\,q^2 = 2\,(\mathrm{S}\,q)^2, \qquad (\mathrm{S} - \mathrm{T})\,q^2 = 2\,(\mathrm{V}\,q)^2.$

40. Since, by (24), $\mathrm{V}\,r\,.\,\mathrm{V}\,q = \mathrm{K}\,(\mathrm{V}\,q\,.\,\mathrm{V}\,r)$, (39′) gives by (20),

(52)
$$\begin{aligned} \mathrm{S}rq &= \mathrm{S}r.\mathrm{S}q + \mathrm{S}\,(\mathrm{V}r.\mathrm{V}q) & \mathrm{V}rq &= \mathrm{S}r.\mathrm{V}q + \mathrm{S}q.\mathrm{V}r + \mathrm{V}\,(\mathrm{V}r.\mathrm{V}q) \\ &= \mathrm{S}r.\mathrm{S}q + \mathrm{S}\,(\mathrm{V}q.\mathrm{V}r), & &= \mathrm{S}r.\mathrm{V}q + \mathrm{S}q.\mathrm{V}r - \mathrm{V}\,(\mathrm{V}q.\mathrm{V}r). \end{aligned}$$

Equations (39′) and (52) give

(53) $$\mathrm{S}\,q\,r = \mathrm{S}\,r\,q,$$

(54) $\mathrm{V}\,qr + \mathrm{V}\,rq = 2(\mathrm{S}r.\mathrm{V}q + \mathrm{S}q.\mathrm{V}r), \quad \mathrm{V}\,qr - \mathrm{V}\,rq = 2\,\mathrm{V}\,(\mathrm{V}q.\mathrm{V}r).$

41. By the aid of the associative principle, we may extend (8) and (15) to

(55) $\qquad \mathrm{K}\,\Pi = \Pi'\,\mathrm{K}, \qquad$ and $\quad (\Pi q)^{-1} = \Pi'\,q^{-1},$

in which Π' is used to denote a product composed of the same fac-

tors as Π, but taken in the reverse order. Equations (21) give by (55)

(56) $\quad 2\,\mathrm{S}\,\Pi = (1+\mathrm{K})\,\Pi = \Pi + \Pi'\mathrm{K}, \quad 2\,\mathrm{V}\,\Pi = \Pi - \Pi'\mathrm{K}.$

From equations (55) and (56) may be deduced, by means of (20), for the *products of vectors*;

(57) $$\mathrm{K}\,\Pi = \pm\,\Pi',$$

(58) $\quad \mathrm{S}\,\Pi = \tfrac{1}{2}(\Pi \pm \Pi') = \pm\,\mathrm{S}\,\Pi', \quad \mathrm{V}\,\Pi = \tfrac{1}{2}(\Pi \mp \Pi') = \mp\,\mathrm{V}\,\Pi',$

the upper signs being used when the number of factors in Π is even, and the under ones when it is odd.

Equations (39′) may, by means of (19 a) and (24), be thus deduced as a special case of (56),

$$\begin{aligned} 2\,\mathrm{S}\,qr &= qr + \mathrm{K}r.\mathrm{K}q = (\mathrm{S}q+\mathrm{V}q)(\mathrm{S}r+\mathrm{V}r) + (\mathrm{S}r-\mathrm{V}r)(\mathrm{S}q-\mathrm{V}q) \\ &= 2\mathrm{S}q.\mathrm{S}r + \mathrm{V}q.\mathrm{V}r + \mathrm{V}r.\mathrm{V}q = 2\mathrm{S}q.\mathrm{S}r + (1+\mathrm{K})(\mathrm{V}q.\mathrm{V}r) \\ &= 2\,\mathrm{S}q.\mathrm{S}r + 2\,\mathrm{S}(\mathrm{V}q.\mathrm{V}r); \end{aligned}$$

and similarly

$$2\,\mathrm{V}\,qr = qr - \mathrm{K}r.\mathrm{K}q = 2\,\mathrm{S}q.\mathrm{V}r + 2\,\mathrm{S}r.\mathrm{V}q + 2\,\mathrm{V}(\mathrm{V}q.\mathrm{V}r).$$

42. By means of (36) and (34′), equation (53) may be transformed to

$$\mathrm{T}q\,.\,\mathrm{T}r\,.\cos\angle qr = \mathrm{T}q\,.\,\mathrm{T}r\,.\cos\angle rq;\ \text{whence}$$

(59) $$\angle qr = \pm\,\angle rq.$$

Whence it follows that $\sin\angle qr = \pm\sin\angle rq$, and therefore, by (34″),

(60) $$\mathrm{T\,V\,U}\,qr = \mathrm{T\,V\,U}\,rq.$$

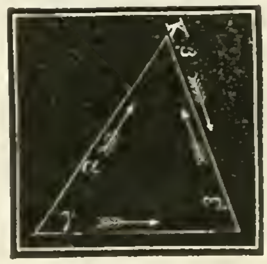
Using P to denote a product composed of the same factors as Π, but altered in their arrangement by any cyclical permutation, equations (53), (59), and (60) may be generalized as follows:

(61) $\quad \mathrm{S}\,\Pi = \mathrm{S\,P}, \quad \angle\Pi = \pm\angle\mathrm{P}, \quad \mathrm{T\,V\,U}\,\Pi = \mathrm{T\,V\,U\,P}.$

Taking the tensor of (36), we find $\mathrm{T\,V}\,q = \mathrm{T}q\,.\,\mathrm{T\,V\,U}\,q$; whence

(62) $$\mathrm{T\,V}\,\Pi = \mathrm{T\,V\,P}.$$

NOTE. — The following simple applications of quaternion analysis are appended by way of examples.



I. Let the vectors α, β, γ form the three sides of a triangle, their positive directions being indicated by the arrows in the figure.

By § 3, $$\gamma = \alpha - \beta;$$

therefore $$\gamma^2 = \alpha^2 - \alpha\beta - \beta\alpha + \beta^2,$$

by (17) and (24), $$= -\mathrm{T}\gamma^2 = -\mathrm{T}\alpha^2 - (1 + \mathrm{K})\,\alpha\beta - \mathrm{T}\beta^2;$$

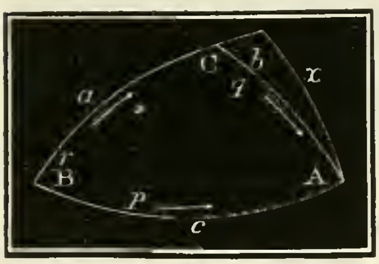
by (21), $$\mathrm{T}\gamma^2 = \mathrm{T}\alpha^2 + 2\,\mathrm{S}\,\alpha\beta + \mathrm{T}\beta^2;$$

by (31) and (33), $$= \mathrm{T}\alpha^2 + 2\,\mathrm{T}\,\alpha\beta \,.\, \cos\frac{\alpha}{\beta^{-1}} + \mathrm{T}\beta^2$$

$$= \mathrm{T}\alpha^2 + \mathrm{T}\beta^2 - 2\,\mathrm{T}\alpha \,.\, \mathrm{T}\beta \,.\, \cos\frac{\alpha}{\beta};$$

but, as $\mathrm{T}\alpha$, &c., represent the lengths of the sides of the triangle, this equation expresses the well-known relation $c^2 = a^2 + b^2 - 2\,a\,b\cos \mathrm{C}$.

II. Let the *versor quaternions* p, q, r form the three sides of a spherical triangle, (*see note to* § 18), the positive directions of their angles being indicated by the arrows in the figure. If the usual notation be also employed for the sides and angles, we shall have



$$\mathrm{C} = \frac{\mathrm{Ax.}\,r^{-1}}{\mathrm{Ax.}\,q}, \quad a = \angle r, \quad b = \angle q, \quad c = \angle p.$$

A, B, and C, may also be employed to denote the directions from the centre of the sphere to the respective vertices of these angles.

By § 9, $$p = q\,r;$$

by (39′), $$\mathrm{S}p = \mathrm{S}q \,.\, \mathrm{S}r + \mathrm{S}(\mathrm{V}q \,.\, \mathrm{V}r),$$

by (31) and (33), $$= \mathrm{S}q \,.\, \mathrm{S}r + \mathrm{T}(\mathrm{V}q \,.\, \mathrm{V}r)\cos\frac{\mathrm{Ax.}\,q}{\mathrm{Ax.}\,r^{-1}};$$

similarly, $$\mathrm{V}p = \mathrm{S}q \,.\, \mathrm{V}r + \mathrm{S}r \,.\, \mathrm{V}q + \mathrm{V}(\mathrm{V}q \,.\, \mathrm{V}r),$$

by (33 a), $$= \mathrm{S}q \,.\, \mathrm{V}r + \mathrm{S}r \,.\, \mathrm{V}q + \mathrm{T}(\mathrm{V}q \,.\, \mathrm{V}r)\sin\frac{\mathrm{Ax.}\,q}{\mathrm{Ax.}\,r^{-1}} \,.\, \sqrt{-1}_{\mathrm{C}}.$$

By (34′) and (34″), the first of these two equations is equivalent to the well-known *fundamental equation* of spherical trigonometry,

$$\cos c = \cos a \cos b + \sin a \sin b \cos \mathrm{C};$$

and the second is equivalent to

$$(m)\quad \sin c \,.\, \sqrt{-1}_p = \cos b \sin a \,.\, \sqrt{-1}_r + \cos a \sin b \,.\, \sqrt{-1}_q - \sin a \sin b \sin \mathrm{C} \,.\, \sqrt{-1}_{\mathrm{C}}.$$

This last equation, being the value of a vector, involves, by § 27, three independent elements, and is therefore equivalent to three *independent* equations. These might be obtained by projection in any three mutually rectangular directions, as in § 27. We

will develop the projections in the directions of Ax. q and A; and for this purpose we must multiply each term of (m) by the cosine of the angle between the vector involved in that term and the direction of Ax. q or A. The angle between the direction A and the plane of r being denoted by x, the multipliers of these terms are respectively

for the direction Ax. q, the cosines of the angles A, $180° - C$, $0°$, $90°$;
" " A " " " $90°$, $90° - x$, $90°$, b.

The two equations thus derived from (m) are therefore

$$\sin c \cos A = -\cos b \sin a \cos C + \cos a \sin b,$$

$$0 = \cos b \sin a \sin x - \sin a \sin b \sin C \cos b.$$

The first of these is another *fundamental equation*, and the second gives, by dividing by $\sin a \cos b$, simply the equation

$$\sin x = \sin b \sin C.$$

PROPERTIES OF CURVATURE IN THE ELLIPSE AND HYPERBOLA.

By Chauncey Wright, Nautical Almanac Office, Cambridge, Mass.

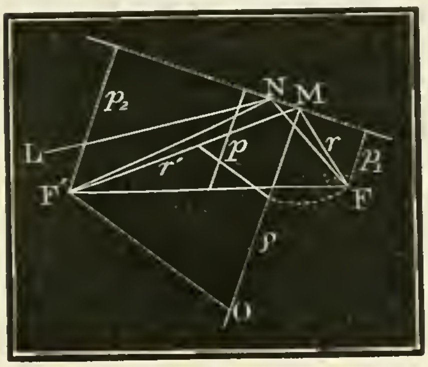
The equation of curvature, $\varrho = \frac{r\,r'}{p}$, (which, as furnishing a direct and simple means of constructing points in the evolute of the ellipse or hyperbola by the method of the fourth proportional, was published in No. X. Vol. I. of this Journal, among the Prize Problems,) can itself be easily deduced from a simple geometrical construction.

1. In this equation r and r' denote the radii vectores from the foci to any point of the curve; p denotes the perpendicular line from the centre to the tangent of this point, and ϱ the radius of curvature.

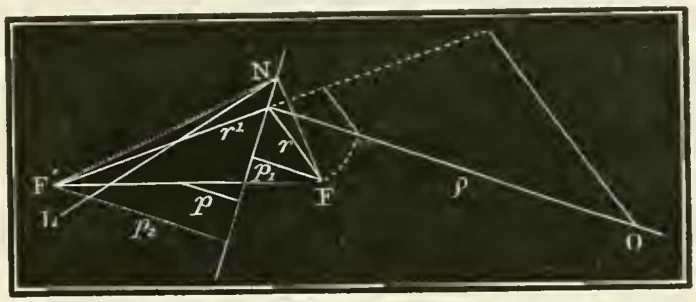
Further, let p_1 and p_2 denote the perpendicular lines from the foci to the tangent, and let ε denote the angle formed by both the radii with the tangent; A and B the semi-axes of the curve, and φ the angle formed by the radius r with the axis A.

If to two points, M and N, of an ellipse or an hyperbola, distant from each other by the element of arc $d\,s$, radii be drawn from both foci, and if the angle included by the radii from the first focus F be

denoted by $d\varphi$, and if a tangent be drawn through one of these points M, and the radius from the first focus F to the other point N be extended to this tangent (an infinitesimal distance of the second order), and from the point of intersection the line NL be drawn, forming the same angle with the tangent as the intersecting radius; then the angle included between this line NL and the radius r' from the second focus F' to the first point M, will be $d\varphi$, and the distance apart of these two lines at the second focus F' will be in the ellipse, $(r + r')\,d\varphi$, and in the hyperbola $(r' - r)\,d\varphi$, or in both $2\,A\,d\varphi$.



ELLIPSE.



HYPERBOLA.

The angle which the line NL forms with the other radius NF' from the second focus is obviously equal to twice the angle which the tangents of the two points M and N form with each other, and we may express it by $2\,d\tau$. The distance apart of these lines at the second focus is therefore $2\,r'\,d\tau$; hence the distances $2\,A\,d\varphi$ and $2\,r'\,d\tau$ are equal or

$$A\,d\varphi = r'\,d\tau;$$

but

$$r\,d\varphi = d\,s \sin\varepsilon,$$

and dividing the latter equation by the former we have

$$\frac{r}{A} = \frac{d\,s \sin\varepsilon}{r'\,d\tau}, \text{ hence } \frac{r\,r'}{A \sin\varepsilon} = \frac{d\,s}{d\tau} = \varrho, \text{ the radius of curvature.}$$

Since from the figures $p_1 = r \sin\varepsilon$ and $p_2 = r' \sin\varepsilon$; p, which is the arithmetical mean of p_1 and p_2, is in the ellipse equal to $\frac{r' + r}{2} \sin\varepsilon$, and in the hyperbola $\frac{r' - r}{2} \sin\varepsilon$, or in both

$$p = A \sin \varepsilon;$$

hence
$$\varrho = \frac{r\,r'}{p}.$$

The construction of the centre of curvature O is given in the figures.

2. The equation of curvature $\varrho = \frac{A^2 B^2}{p^3}$ may be deduced from the one above by the theorem,

$$p_1 p_2 = B^2,$$

which is easily proved as follows.

The angle included by the radii vectores to any point is in the ellipse the supplement of $2\,\varepsilon$ and in the hyperbola it is equal to $2\,\varepsilon$. If now the distance between the foci be denoted by $2\,C$ we have by trigonometry

$$4\,C^2 = r^2 \pm 2\,r\,r' \cos 2\,\varepsilon + r'^2,$$

(the upper sign for the ellipse and the lower sign for the hyperbola) and since $\cos 2\,\varepsilon = 1 - 2\sin^2\varepsilon$ we have

$$4\,C^2 = r^2 \pm 2\,r\,r' + r'^2 \mp 4\,r\,r' \sin^2\varepsilon = (r \pm r')^2 \mp 4\,r\,r' \sin^2\varepsilon = 4\,A^2 \mp 4\,r\,r' \sin^2\varepsilon;$$

hence
$$r\,r' \sin^2\varepsilon = \pm (A^2 - C^2) = B^2 = p_1 p_2.$$

$$\therefore r\,r' = \frac{B^2}{\sin^2\varepsilon} \text{ and } \frac{r\,r'}{A \sin\varepsilon} = \frac{B^2}{A \sin^3\varepsilon} = \frac{A^2 B^2}{p^3} = \varrho.$$

3. The mechanical properties of the ellipse and hyperbola may be easily deduced from the equations

$$\varrho = \frac{A^2 B^2}{p^3} = \frac{B^2}{A \sin^3\varepsilon} = \frac{r\,r'}{p} = \frac{r\,r'}{A \sin\varepsilon},$$

in the two cases of central forces for which these curves are the paths described.

First, if the centre of attraction or repulsion be the centre of the ellipse or hyperbola, and if v denote the velocity with which the material point is at any time, t, moving, and r_c the radius vector of this point from the centre, ε_c the angle which it forms with the tan-

gent and φ_c the angle which it forms with the axis A; the principle of *equal areas* may be expressed by the equation,

$$p\,v = a = \text{a constant},$$

or putting $p = r_c \sin \varepsilon_c$ and $v = \frac{d\,s}{d\,t}$, and if $d\,\sigma =$ the elementary area described by the radius r_c in the time $d\,t$ we have

$$r_c \sin \varepsilon_c \times \frac{d\,s}{d\,t} = a = \frac{r_c^2\,d\,\varphi_c}{d\,t} = \frac{2\,d\,\sigma}{d\,t}.$$

If the area and time be reckoned from the same origin, and if T denote the time of one complete revolution, a is equal to twice the whole area of the ellipse divided by T or $a = \frac{2\,A\,B\,\pi}{T}$.

If we substitute in the equation of *living force*

$$d\,(v^2) = 2\,R\,d\,r_c$$

(in which R is the force of attraction or repulsion) the value of v from the equation of equal areas $v = \frac{a}{p}$ we get

$$d\left(\frac{a^2}{p^2}\right) = -\frac{2\,a^2}{p^3}\,d\,p = 2\,R\,d\,r_c,$$

and hence $$R\,\frac{d\,r_c}{d\,p} = -\frac{a^2}{p^3}.$$

But in general $$\varrho = r_c\,\frac{d\,r_c}{d\,p};\text{*}$$

hence in the ellipse and hyperbola

$$r_c\,\frac{d\,r_c}{d\,p} = \frac{A^2\,B^2}{p^3} = -\frac{a^2}{R\,p^3}\,r_c;$$

hence $$R = -\frac{a^2}{A^2\,B^2}\,r_c:$$

or the force is proportional to the distance from the centre.

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* It is readily seen by the construction of these quantities that $d\,p_x = r_x \cos \varepsilon_x\,d\,\tau$, $d\,r_x = d\,s \cos \varepsilon_x$, and hence the ratio $\frac{d\,r_x}{d\,p_x} = \frac{d\,s}{r_x\,d\,\tau} = \frac{\varrho}{r_x}$; in which p_x, r_x, and ε_x, are the perpendicular to the tangent, the radius vector and its angle with the tangent, for any centre and any curve whatever.

Since in the ellipse $a = \frac{2\,A\,B\,\pi}{T}$ we get $R = -\frac{4\,A^2\,B^2\,\pi^2}{A^2\,B^2\,T^2}\,r_c = -\frac{4\,\pi^2}{T^2}\,r_c$; hence the time of revolution is independent of the magnitude or form of the ellipse. The small oscillations of a free pendulum are therefore isochronous.

Secondly, when one of the foci F of the ellipse or hyperbola is the centre of force, we have the mechanical equations

$$v = \frac{a}{p_1} \text{ and } d(v^2) = 2\,R\,d\,r = -\frac{2\,a^2}{p_1^{\,3}}\,d\,p_1,$$

and the equations of curvature

$$\varrho = r\,\frac{d\,r}{d\,p_1} = \frac{B^2}{A\,\sin^3\varepsilon};$$

hence $R\,\frac{d\,r}{d\,p_1} = -\frac{a^2}{p_1^{\,3}} = -\frac{a^2}{r^3\,\sin^3\varepsilon} = R\,\frac{B^2}{A\,r\,\sin^3\varepsilon}$ and $R = -\frac{A\,a^2}{B^2\,r^2}$;

the force is therefore inversely proportional to the square of the distance, and in the ellipse, since $a = \frac{2\,A\,B\,\pi}{T}$,

$$R = -\frac{4\,A^3\,B^2\,\pi^2}{T^2\,B^2\,r^2} = -\frac{4\,A^3\,\pi^2}{T^2\,r^2}.$$

The time of revolution is therefore independent of the minor axis B, and its square is proportional to the third power of the major axis, according to Kepler's third law.

The general equation

$$R = -\frac{a^2\,A\,r_x\,\sin^3\varepsilon}{B^2\,p_x^{\,3}}$$

includes the two cases above, and all cases in which r_x and p_x are the radius and perpendicular to the tangent from any centre of force whatever.

If we substitute in it the radius and perpendicular from the centre, r_c and $p = A\,\sin\varepsilon$, or the radius and perpendicular from the

focus, r and $p_1 = r \sin \varepsilon$, the force R becomes independent of direction, and is a function of the radius only.

If we resume the mechanical equations

$$r = \frac{a}{p_x} \text{ and } d(v^2) = 2 R\, d r_x = \frac{2 a^2}{p_x^3}\, d p_x,$$

and the equation of curvature

$$\frac{\varrho}{r_x} = \frac{d r_x}{d p_x},$$

we have in general

$$\frac{R \varrho}{r_x} = -\frac{a^2}{p_x^3} = -\frac{v^2}{p_x};$$

hence

$$R \frac{p_x}{r_x} = -\frac{v^2}{\varrho}.$$

But $R \frac{p_x}{r_x} = R \sin \varepsilon_x$ is the component of R perpendicular to the path of the moving point, and is equal and opposite to the centrifugal force of this point. Hence this centrifugal or normal force $N = - R \sin \varepsilon_x = \frac{v^2}{\varrho}$. When the centre of force is the centre of the ellipse or hyperbola, $N = \frac{v^2}{\varrho} = m\, r_c \sin \varepsilon_c = m p$ (m being the force at the distance unity from this centre). But from the equation of curvature $\varrho = \frac{r r'}{p}$ we have $v^2 = m p \varrho = m r r'$.

When the centre of force is the focus F of the ellipse or hyperbola we have $N = \frac{v^2}{\varrho} = \frac{m \sin \varepsilon}{r^2}$, and from the equation of curvature $\varrho = \frac{r r'}{A \sin \varepsilon}$ we have $v^2 = \varrho \frac{m \sin \varepsilon}{r^2} = \frac{m r'}{A r}$.

The square of the velocity is therefore proportional in the first case to the product of the radii from the foci; and in the second case to their ratio. Moreover, it is obvious that in the second case the product of any two velocities in two positions at equal and opposite distances from the minor axis is the constant $\frac{m}{A}$; that is,

one velocity is as many times greater than the constant $\sqrt{\frac{m}{A}}$, as the other is less.

Since we have by geometry $r^2 + r'^2 = 2\, r_c^2 + 2\, C^2$, we find

$$4\, A^2 = (r + r')^2 = r^2 + 2\, r\, r' + r'^2 = 2\, r_c^2 + 2\, C^2 + 2\, r\, r';$$

hence $$r\, r' = 2\, A^2 - C^2 - r_c^2 = A^2 + B^2 - r_c^2,$$

and $$v^2 = m\,(A^2 + B^2 - r_c^2).$$

In the second case we have

$$v^2 = \frac{m\, r'}{A\, r} = m\,\frac{2\, A - r}{A\, r} = m\left(\frac{2}{r} - \frac{1}{A}\right).$$

These values of v^2 may also be deduced, though not so readily, by integration, and the determination of constants from the fundamental equation $d(v^2) = 2\, R\, d\, r_x$.

LAW OF GRAVITY.

Problem in Celestial Mechanics.

Prove that the Newtonian law of gravity is such, that the scale of spaces and the scale of times, in the motions of the heavenly bodies, are independent of each other; that is, prove that in two systems similar in construction, but different in scale, all the corresponding motions will be synchronous, whether in the bodies themselves, or in bodies on their surfaces.

PROFESSOR ENCKE'S METHOD OF COMPUTING SPECIAL PERTURBATIONS.*

After the choice of the magnitude of the intervals and of the time of the beginning is settled, either by general considerations, or by a preliminary calculation, the computation arranges itself into the five following divisions: —

1. The computation of the places of the disturbing planet for the assumed times, and their reduction to the values L', r', Ω', i', adopted in the formulæ.

2. The computation of the places of the disturbed planet, and of the values necessary to form the coefficients in the differential equations.

3. The computation of the amount of the disturbing forces in the directions R, S, W.

4. Their substitution in the equations of condition, and the calculation of the values of the differential coefficients themselves.

5. The integration of the latter, or the construction of the table, which exhibits the values of the general integral combined with the different constants.

Our planetary tables give directly the longitude in the orbit, and the radius vector, and therefore the values denoted by L', r', Ω', and i'. So that if the tables are immediately resorted to, by adhering to these values, the perturbations in latitude only will be neglected, which are always so small that their entire omission is of no importance. In this case, it will be necessary to reduce the lon-

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* Jahrbuch, 1838. Translated from the German for the Use of the American Ephemeris and Nautical Almanac, by Charles Henry Davis, Commander United States Navy, Superintendent of the Nautical Almanac.

gitudes collectively, by the addition of the precession, to a fixed mean equinox, usually that of the time of the beginning.

If, on the contrary, the places are interpolated from the ephemeris which give the longitudes in the ecliptic, and the latitudes, it will be necessary to make use of the latter with rigorous exactness in order to find ☊$'$ and i', and thence to deduce the values of L', the longitudes in the orbit. If l' and b' are the longitudes and latitudes already reduced to the fixed mean equinox, then the nutation with the opposite sign and the precession are already applied, and it will be necessary to compute from two values l_0' and l_1', b_0' and b_1', selected as advantageously as the circumstances will allow, the following formulæ: —

$$\sin\left(\tfrac{1}{2}(l_1' + l_0') - \Omega'\right) \tan i' = \frac{\sin(b_1' + b_0')}{2\cos b_1' \cos b_0' \cos\frac{1}{2}(l_1' - l_0')};$$

$$\cos\left(\tfrac{1}{2}(l_1' + l_0') - \Omega'\right) \tan i' = \frac{\sin(b_1' - b_0')}{2\cos b_1' \cos b_0' \sin\frac{1}{2}(l_1' - l_0')};$$

$$L' = l' + \tan\tfrac{1}{2}i'^2 \sin 2(l' - \Omega') + \tfrac{1}{2}\tan\tfrac{1}{2}i'^4 \sin 4(l' - \Omega') \ldots .$$

the last formula for every l'; from the first two are obtained ☊$'$ and i', including the perturbations in latitude as accurately as the tables permit. For all the older planets the term containing $\tan\frac{1}{2}i'^4$ is nearly insensible. In this manner the first part of the computation is given in the desired form.

For the second, or the place of the disturbed planet, it will be convenient to separate, in the expression of the differential coefficients, the quantities which, depending merely upon the elements remain constant during the greater number of intervals, from the other quantities varying with the time. If we designate by R_0, S_0, W_0, for the present, what has been denoted in the equations (23) by $\frac{k R_0}{\sqrt{p}}$, $\frac{k S_0}{\sqrt{p}}$, $\frac{k W_0}{\sqrt{p}}$, and if we call the amount of one interval expressed in mean days ω, observing, moreover, that for the purpose of integration the differential coefficients must be multiplied by ω, with the

exception of $\frac{d\mu}{dt}$, which on account of the double integral must be multiplied by ω^2, and if we introduce for the constant factors the notation (1), (2), (3), &c., we shall have the following formulæ: —

$(1) = \frac{k\,m'}{\sqrt{p}}\,\omega$, where m' is expressed in seconds, or $\log m' = 5.3144251$ + Brigg's log of the mass of the disturbing planet in parts of the sun's mass, and $\log k = 8.2355814$;

$$(2) = \frac{1}{\sin i}; \qquad (3) = a\cos\varphi; \qquad (4) = \frac{p}{e};$$

$$(5) = \frac{1}{e}; \qquad (6) = \tan\tfrac{1}{2}i; \qquad (7) = \frac{3\,k\,\omega}{\sqrt{a}}\,e;$$

$$(8) = \frac{3\,k\,\omega}{\sqrt{a}}\,p; \qquad (9) = p\tan\tfrac{1}{2}\varphi; \qquad (10) = 2\cos\varphi;$$

$$(11) = \tan\tfrac{1}{2}\varphi; \qquad (12) = p\cot\varphi; \qquad (13) = \cot\varphi.$$

The equations (22) and (23) are

$$(22) \qquad R_0 = (1)\,R'; \qquad S_0 = (1)\,S'; \qquad W_0 = (1)\,W';$$

$$\omega^2\frac{d\mu}{dt} = -\frac{3k}{\sqrt{a}}\,e\sin v\,R_0 - \frac{3k}{\sqrt{a}}\cdot\frac{p}{r}\cdot S_0;$$

$$\omega\frac{dM}{dt} = (2\,r\cos\varphi - p\cot\varphi\cos v)\,R_0$$
$$- (p+r)\cot\varphi\sin v\,S_0 + \omega\int\frac{d\mu}{dt}\cdot dt;$$

$$(23) \qquad \omega\frac{d\varphi}{dt} = a\cos\varphi\sin v\,R_0 + a\cot\varphi\left(\frac{p}{r} - \frac{r}{a}\right)S_0;$$

$$\omega\frac{d\pi}{dt} = -\frac{p\cos v}{e}\cdot R_0 + \frac{p+r}{e}\sin v\,S_0 + (1-\cos i)\,\omega\frac{d\Omega}{dt};$$

$$\omega\frac{d\Omega}{dt} = \frac{r\sin(v+\pi-\Omega)}{\sin i}\cdot W_0;$$

$$\omega\frac{di}{dt} = r\cos(v+\pi-\Omega)\cdot W_0.$$

Or, if we introduce L in the place of M,

$$\omega\frac{dL}{dt} = -(2\,r\cos\varphi + p\tan\tfrac{1}{2}\varphi\cos v)\,R_0$$
$$+ (p+r)\tan\tfrac{1}{2}\varphi\sin v\,S_0 + (1-\cos i)\,\omega\frac{d\Omega}{dt} + \omega\int\frac{d\mu}{dt}\cdot dt.$$

With the above factors, the equations (23) will be written as follows: —

$$\omega \frac{d i}{d t} = r \cos u \, W_0 ; \qquad \cos u = \cos(v + \pi - \Omega).$$

$$\omega \frac{d \Omega}{d t} = (2)\, r \sin u \, W_0 ;$$

$$\omega \frac{d \varphi}{d t} = (3) \sin v \, R_0 + (3)(\cos v + \cos E)\, S_0 ;$$

$$\omega \frac{d \pi}{d t} = -(4) \cos v \, R_0 + (5)\left(\frac{p}{r} + 1\right) r \sin v \, S_0 + (6)\, r \sin u \, W_0 ;$$

$$\omega^2 \frac{d \mu}{d t} = -(7) \sin v \, R_0 - (8) \frac{1}{r} S_0 ;$$

$$\omega \frac{d L}{d t} = \{-(9) \cos v - (10)\, r\}\, R_0 + (11)\left(\frac{p}{r} + 1\right) r \sin v \, S_0$$
$$+ (6)\, r \sin u \, W_0 + \omega \int \frac{d \mu}{d t}\, d t ;$$

$$\omega \frac{d M}{d t} = \{+(12) \cos v - (10)\, r\}\, R_0 - (13)\left(\frac{p}{r} + 1\right) r \sin v \, S_0$$
$$+ \omega \int \frac{d \mu}{d t}\, d t.$$

These formulæ are wholly identical with (23), except this small change, that, since

$$\frac{p}{r} = 1 + e \cos v, \qquad \frac{r}{a} = 1 - e \cos E,$$

$e(\cos v + \cos E)$ is written instead of $\frac{p}{r} - \frac{r}{a}$, in consequence of which the last term in $\frac{d \varphi}{d t}$ is changed to

$$a \cos \varphi (\cos v + \cos E).$$

It is preferable in numerical computation to add together two small quantities which have with few exceptions the same signs, rather than to subtract from each other two larger quantities of

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Note. — The equations (22) and (23) are the final equations for the perturbations of the elements in the theoretical investigation.

which the sign is always the same. This addition will be particularly facilitated by the Gaussian table of logarithms of sums and differences, which table can be frequently used with great advantage in this calculation. On account of using this table, $\left(\frac{p}{r} + 1\right) r \sin v$ is written instead of $(p + r) \sin v$. If $p > r$, and $\frac{p}{r}$ therefore an improper fraction, for A in the Gaussian table taken as the $\log \frac{p}{r}$ the adjoining logarithm in the column C is equal to $\log \left(\frac{p}{r} + 1\right)$; and again, if $p < r$, or $\frac{p}{r}$ is a proper fraction, the column B gives the $\log \left(\frac{p}{r} + 1\right)$ for $A = \log \frac{r}{p}$; thus we enter always A with the difference of the logarithms of p and r, and find then either in B or C the $\log \left(\frac{p}{r} + 1\right)$ according as the latter is smaller or greater than the logarithm of the number two, which can be immediately seen.

We have, therefore, upon the whole, to compute $\sin v$, $\cos v$, $\cos E$, $\sin u$, $\cos u$, $\log r$, of which the coefficients are very simple combinations. For this purpose different formulæ may be applied. Those to which I have been accustomed are the following: —

If the elements are given in the form suited to computation, and thence, also, the quantities for a certain time T, from which the perturbations are to be computed, referred to a fixed mean equinox, the same as that adopted for the disturbing planet, which quantities are,

L, the mean longitude of the disturbed planet,
μ, the mean daily sidereal motion,
π, the longitude of the perihelion,
φ, the angle, the sine of which is equal to the eccentricity,
Ω, the longitude of the ascending node,
i, the inclination,

then we compute in the first place the constants,

$a = \left(\frac{k}{\mu}\right)^{\frac{2}{3}}$, in which the $\log k$, since μ is given in seconds, here $=$ 3.5500066,

$e'' = \sin\varphi$ in seconds, or the logarithm of $206265 = 5.3144256$ added to $\log e$,

$\left.\begin{array}{l}\sqrt{a}\,(1-e)\\ \sqrt{a}\,(1+e)\end{array}\right\}$ from which occurs the check, that $\log\sqrt{a}\,(1-e) - \log\sqrt{a}\,(1+e) = \log\tan(45^\circ - \frac{1}{2}\varphi)$,

$p = a\cos^2\varphi$,

and also the above-mentioned constants (1), (2), ... (13).

In order to combine all the constants, we can add here at once the reduction, necessary for the computation of the forces, of the orbit of the disturbing planet to that of the disturbed, or the computation of the quantities I, N, N', by means of the formulæ,

$$
(20)\quad
\begin{aligned}
\sin\tfrac{1}{2}I\sin\tfrac{1}{2}(N+N') &= \sin\tfrac{1}{2}(\Omega'-\Omega)\sin\tfrac{1}{2}(i'+i),\\
\sin\tfrac{1}{2}I\cos\tfrac{1}{2}(N+N') &= \cos\tfrac{1}{2}(\Omega'-\Omega)\sin\tfrac{1}{2}(i'-i),\\
\cos\tfrac{1}{2}I\sin\tfrac{1}{2}(N-N') &= \sin\tfrac{1}{2}(\Omega'-\Omega)\cos\tfrac{1}{2}(i'+i),\\
\cos\tfrac{1}{2}I\cos\tfrac{1}{2}(N-N') &= \cos\tfrac{1}{2}(\Omega'-\Omega)\cos\tfrac{1}{2}(i'-i),
\end{aligned}
$$

where another check occurs in the agreement of $\sin\frac{1}{2}I$ and $\cos\frac{1}{2}I$.

If the computation of the perturbations first begins in general for T, it is necessary to calculate the places for $T - \frac{5}{2}\omega$, $T - \frac{3}{2}\omega$, $T - \frac{1}{2}\omega$, $T + \frac{1}{2}\omega$, ... $T + (i + \frac{1}{2})\omega$, if we wish to have the most convenient integration for the simple integrals that occur most frequently. With a suitable magnitude of the intervals, there is no occasion to go farther back than $T - \frac{5}{2}\omega$. If the slight inconvenience of the preparation of the first constants is not regarded, the places for $T - 3\omega$, $T - 2\omega$, $T - \omega$, T, $T + \omega$, ... $T + i\omega$ may also be calculated. But if the new computation for a later time T' joins on to a previous one, we seek first for the system of elements which, from the earlier computation, comes as near as possible to T', and calculate always with the new system the last place of the previous computation anew, in order to have the means of

ascertaining whether the elements in the former computation have not been regarded as constant for too long a time.

For the calculation of v, r, &c., the following formulæ answer then for every place: —

$$M = L - \pi + \mu\,(t - T);$$
$$M = E - e'' \sin E;$$
$$\sin \tfrac{1}{2} v \sqrt{r} = \sin \tfrac{1}{2} E \sqrt{a\,(1 + e)};$$
$$\cos \tfrac{1}{2} v \sqrt{r} = \cos \tfrac{1}{2} E \sqrt{a\,(1 - e)}.$$

For the solution of the transcendental equation which gives E by means of M, let there be taken any, the nearest possible value of $E \ldots\ldots E'$, and compute

$$M' = E' - e'' \sin E';$$

then is

$$M - M' = E - E' - e''\,(\sin E - \sin E')$$
$$= (E - E')\,(1 - e \cos E'),$$

provided $E' - E$ is regarded as a small quantity of the first order, and those of the second order are neglected for a time, so that a new approximate value of E is

$$E' + \frac{M - M'}{1 - e \cos E'},$$

with which we proceed in the same manner until the true value is obtained. It is almost always unnecessary to repeat the calculation of $1 - e \cos E'$. According to rule, if the first E' was not too much out of the way, the first computed value of $1 - e \cos E'$ might be retained in all the trials.

This process, which appears to be the most rapid and safe for the first approximation, is wholly identical with the rule of Gauss, to make use of the logarithmic difference with $\sin E \ldots\ldots \lambda, \ldots\ldots$ and of the logarithmic difference with the number $e'' \sin E \ldots\ldots \mu, \ldots\ldots$

both naturally referred to the same unit, and then to adopt as the consequent approximate value

$$E' + (M - M') \cdot \frac{\mu}{\mu \pm \lambda}.$$

For, since λ is nothing else than

$$\lambda = \frac{d \log \sin E}{d\, E} = \frac{\cos E}{\sin E},$$

if we omit the modulus of BRIGG'S system, which afterwards disappears of itself, and

$$\mu = \frac{d \log (e'' \sin E)}{d\, (e'' \sin E)} = \frac{1}{e \sin E},$$

therefore is

$$\frac{\mu}{\mu - \lambda} = \frac{1}{1 - \frac{\lambda}{\mu}} = \frac{1}{1 - e \cos E},$$

so that the double sign $\mu \pm \lambda$ is always to be so taken as to agree with the sign of $\cos E$, if λ is always regarded as positive. Both formulæ would be wholly identical, if in practice the uncertainty of the logarithmic differences, in case the approximation is not yet very great, did not make the last form somewhat more doubtful than the first. The latter form, however, will be used with advantage in the last trial, in order to produce an agreement up to the last place of logarithms. Moreover, if several consecutive places are computed, the trials by differences formed from the previous results, and carried on for the new date, are so abridged, that the truth is found after four or five places almost without any trial. On this account, it will not be advisable to exhibit the trials in the actual computation. Only the final result, and its verification, need be given here.

The addition of the constant logarithms $\sqrt{a}\,(1 + e)$, &c., is made in the head by means of a paper, held over the other logarithms, on the lower border of which the constant logarithm stands; and so

in general with every combination of this sort of constants with variables.

The formation of the other quantities,

$$u = v + \pi - \Omega,$$

$$r \sin u,\ \ r \cos u,\ \ \sin v,\ \ \cos v,\ \ (\cos E + \cos v),\ \left(\frac{p}{r} + 1\right),\ \ r \sin v,$$

requires no explanation. The latter quantity, $r \sin v$, supplies a convenient check, because

$$r \sin v = a \cos \varphi \sin E,$$

and differs therefore by a constant logarithm from the $\log \sin E$ already written down.

If we wish to compute together $\frac{dM}{dt}$ and $\frac{dL}{dt}$ for the sake of verification, one value being otherwise sufficient, the Gaussian logarithms are not to be applied for

$$-(2\, r \cos \varphi - p \cot \varphi \cos v),$$
$$-(2\, r \cos \varphi + p \tan \tfrac{1}{2} \varphi \cos v),$$

nor for

$$-(10)\, r + (12) \cos v,$$
$$-(10)\, r - (9) \cos v,$$

but the numbers are to be sought in preference. If only one of the two values is computed, still the numbers can be made use of conveniently.

The appropriate combination of these values with the different constants (2) to (13) gives the logarithms of the coefficients of R_0, S_0, W_0, in the differential equations, by which also a portion of the fourth part is completed.

After this follows the third part, the computation of the forces, for which the requisite values N, N', I, have already been found. The collective formulæ are,

$$\sin\beta' = \sin(L' - (\Omega' + N'))\sin I,$$
$$\tan\lambda' = \tan(L' - (\Omega' + N'))\cos I,$$

where $\cos\lambda'$ must always have the same sign as $\cos(L' - (\Omega' + N'))$,

$$z' = r'\sin\beta',$$
$$y' = r'\cos\beta'\sin(\lambda' - (v + \omega')),$$
$$x' = r'\cos\beta'\cos(\lambda' - (v + \omega')),$$

in which $\omega' = \pi - \Omega - N,$

and furthermore, $\cos\beta'\cos(\lambda' - (v + \omega')) = \cos\gamma',$

$$r'\sin\gamma' = \varrho\sin l',$$
$$r - r'\cos\gamma' = r - x' = \varrho\cos l',$$

without actually taking out the angles γ' and l'. Then,

$$\frac{1}{\varrho^3} - \frac{1}{r'^3} = \varDelta,$$

in which the Gaussian logarithms are used with great advantage. Finally,

$$W_0 = (1)\,\varDelta z', \qquad S_0 = (1)\,\varDelta y', \qquad R_0 = (1)\left\{\varDelta x' - \frac{r}{\varrho^3}\right\}.$$

These forces must be combined with the appropriate coefficients; namely, for

$$\begin{array}{lllll}
d i \ldots & r\cos u & \ldots\ldots & \text{with} & W_0, \\
d\Omega \ldots & (2)\, r\sin u & \ldots\ldots & " & W_0, \\
d\varphi \ldots & \begin{cases} (3)\sin v \\ (3)(\cos v + \cos E) \end{cases} & \begin{array}{l}\ldots\ldots \\ \ldots\ldots\end{array} & \begin{array}{l} " \\ " \end{array} & \begin{array}{l} R_0, \\ S_0, \end{array} \\
d\pi \ldots & \begin{cases} -(4)\cos v \\ +(5)\left(\frac{p}{r} + 1\right) r\sin v \\ +(6)\, r\sin u \end{cases} & \begin{array}{l}\ldots\ldots \\ \ldots\ldots \\ \ldots\ldots\end{array} & \begin{array}{l} " \\ " \\ " \end{array} & \begin{array}{l} R_0, \\ S_0, \\ W_0, \end{array} \\
d\mu \ldots & \begin{cases} -(7)\sin v \\ -(8)\,\frac{1}{r} \end{cases} & \begin{array}{l}\ldots\ldots \\ \ldots\ldots\end{array} & \begin{array}{l} " \\ " \end{array} & \begin{array}{l} R_0, \\ S_0, \end{array}
\end{array}$$

$$dL \,.\,. \begin{cases} -(9)\cos v - (10)\, r & \ldots \quad \text{with } R_0, \\ +(11)\left(\frac{p}{r}+1\right) r \sin v & \ldots \quad \text{“} \quad S_0, \\ +(6)\, r \sin u & \ldots \quad \text{“} \quad W_0, \end{cases}$$

$$dM \begin{cases} +(12)\cos v - (10)\, r & \ldots \quad \text{“} \quad R_0, \\ -(13)\left(\frac{p}{r}+1\right) r \sin v & \ldots \quad \text{“} \quad S_0. \end{cases}$$

The double integral

$$\iint \frac{d\mu}{dt}\, dt^2$$

of L and M is added separately after the integration. In this manner are given the numerical values of the differential coefficients, copious examples of the integration of which are contained in the treatise upon mechanical quadratures. The constants to be added for the elements i, ☊, π, φ, are the original adopted values for a fixed time T, and the integration gives immediately the increase, since the factor ω is contained in (1). The first integral gives for μ, $\omega\,\varDelta\,\mu$, because the quadratic factor ω^2 appears in the differential coefficient; ONE multiplication by ω is affected by means of the constant (1), the OTHER by (7) and (8). For L and M, there is still to be added $L_0 + \mu_0\,(t - T)$ and $M_0 + \mu_0\,(t - T)$, besides the double integral, and the result of the integration of $\frac{dL}{dt}$. Finally, it must not be forgotten that all longitudes are referred to a fixed mean equinox, and therefore the precession and nutation must be applied for any other epoch.

So far as the accuracy required in this computation is concerned, logarithms of five places of decimals appear to be sufficient. If the perturbations should be so great, that these are not satisfactory for a fixed interval, it would be expedient, for other reasons, so to reduce the interval that the five decimal places would still afford all that is desired. They are also to be taken in preference, because with them the computation can be carried through very rapidly. With

some effort, the complete computation of the perturbations, for about ten intervals, can be finished in one day, even if, which may be advisable as a general rule, v, r, and all the constants are calculated with six places of decimals, in order to be certain of the final fifth place, and in order not to obtain in the solution of the transcendental equation for E results which, sufficiently accurate in themselves, might still want conformity in the differences by means of which a sure proof of the correctness of the computation is obtained. This verification by means of differences should not be neglected, particularly at the end. The fitting on of the later computation to the previous one affords a security against constant errors; the regularity of the differences, against accidental errors.

In the following detailed example, in which no single number is wanting that was incidentally computed, with the exception of the inconsiderable calculation of Ω' and i', and the trials for E, in which the comparatively small tediousness and difficulty of this computation are plainly evident, I have, for the convenience of printing, carried out the calculation with four decimals.

The comparison of the results obtained with five decimals with these shows that in ordinary cases four decimals will be sufficient. Still, the saving of time does not appear to me to be important enough to sacrifice to it the advantage of obtaining the angles accurate within a few seconds, which is possible with five decimals, while with four decimals parts of minutes only are given.

The previously established elements for June 12, 1836, derived from initial values resulting from a former computation, will answer for an example of the integration.

PERTURBATIONS OF VESTA BY JUPITER.

1836, JULY 3 — DECEMBER 18.

PLACES OF JUPITER.

The interpolation from the Almanac gives for the apparent place : —

1836, 0h. P. M. T.	Longitude ♃	Latitude ♃	Radius Vector.
July 3	115° 58′ 35″.8	+0° 23′ 19″.9	5.25433
August 14	119 23 6.1	27 45.6	5.26916
September 25	122 46 26.2	32 4.0	5.28371
November 6	126 8 40.4	36 14.3	5.29794
December 18	129 29 53.1	40 16.1	5.31179

The subtraction of the nutation and precession since 1810, January 0, in order to bring everything to the mean equinox of this epoch, gives the reduced longitudes, —

1836, 0h. P. M. T.	t — 1810.	Correction for Nutation.	Correction for Precession.	Reduced Longitudes.
July 3	9681	+11″.6	—22′ 11″.3	115° 36′ 36″.1
August 14	9723	10.4	22 17.1	119 0 59.4
September 25	9765	11.3	22 22.9	122 24 14.6
November 6	9807	12.1	22 28.7	125 46 23.8
December 18	9849	10.4	22 34.5	129 7 29.0

From the places,

July	3	115° 36′ 36″.1	+0° 23′ 19″.9
December	18	129 7 29.0	40 16.1

we have, $\Omega' = 98° \; 22' \; 56''.0$ $\qquad i' = 1° \; 18' \; 45''.9$;

whence the reduction to the longitudes in the orbit is

$$+27''.073 \sin 2 \,(\text{reduced longitudes} - \Omega').$$

Consequently we have

1836, 0h. P. M. T.	Reduct.	L'	Log r'
July 3	+15″.3	115° 36′ 51″.4	0.720517
August 14	17.9	119 1 17.3	0.721741
September 25	20.1	122 24 34.7	0.722939
November 6	22.1	125 46 45.9	0.724107
December 18	23.8	129 7 52.8	0.725241

ELEMENTS OF VESTA.

1836, June 12, 0^h Paris Mean Time.

$M = 319^\circ\ 13'\ 1''.8$	$\pi = 250^\circ\ 7'\ 56''.4$
$\mu = 977''.83172$	$\Omega = 102\ 59\ 1.6$
$\varphi = 5^\circ\ 2'\ 33''.6$	$i = 7\ 8\ 15.7$

2.990264	8.943977	9.998316	0.087898
3.550007	5.314425	9.996632	0.036587
0.559743	4.258402	0.369794	0.373162
0.186581	$\log e''$	$\log p$	9.960043
0.373162		$\log \sqrt{a}\,(1+e)$ . . .	0.204875
$\log a$		$\log \sqrt{a}\,(1-e)$ . . .	0.166603

$\log \frac{42\,k}{\sqrt{p}} = 9.673934$ $\log \frac{126\,k}{\sqrt{a}} = 0.149371$

$\log m' = 2.291616$ Jupiter's mass in seconds.

$\log (1) \ldots \frac{42\,k\,m'}{\sqrt{p}} \ldots 1.965550$	$\log (7) \ldots \frac{126\,k}{\sqrt{a}}\,e \ldots 9.093348$
$\log (2) \ldots \frac{1}{\sin i} \ldots 0.905688$	$\log (8) \ldots \frac{126\,k}{\sqrt{a}}\,p \ldots 0.519165$
$\log (3) \ldots a \cos \varphi \ldots 0.371478$	$\log (9) \ldots p \tan \frac{1}{2}\varphi \quad 9.013582$
$\log (4) \ldots \frac{p}{e} \ldots 1.425817$	$\log (10) \ldots 2 \cos \varphi \quad 0.299346$
$\log (5) \ldots \frac{1}{e} \ldots 1.056023$	$\log (11) \ldots \tan \frac{1}{2}\varphi \quad 8.643788$
$\log (6) \ldots \tan \frac{1}{2} i \ldots 8.794967$	

$\Omega' \ldots 98^\circ\ 22'\ 56''.0$	$i' \ldots 1^\circ\ 18'\ 45''.9$	$\frac{1}{2}(\Omega' - \Omega) \ldots 177^\circ\ 41'\ 57''.2$
$\Omega \ldots 102\ 59\ 1.6$	$i \ldots 7\ 8\ 15.7$	$\frac{1}{2}(i' + i) \ldots 4\ 13\ 30.8$
		$\frac{1}{2}(i' - i) \ldots -2\ 54\ 44.9$

8.867333	8.705952_n	$\frac{1}{2}(N + N') = 3^\circ\ 20'\ 3''.3$
8.603635	9.999650_n	$\frac{1}{2}(N - N') = 177\ 42\ 9.0$
9.998818	9.999439	$N \ldots 181\ 2\ 12.3$
7.470968	8.602453	$\pi - \Omega \ldots 147\ 8\ 54.8$
8.705602	9.999089_n	$\omega' \ldots 326\ 6\ 42.5$
9.999264	9.999651	$N' \ldots 185\ 37\ 54.3$
8.706338	9.999438	$\Omega' \ldots 98\ 22\ 56.0$
$\frac{1}{2} I \ldots 2^\circ\ 54'\ 54''.2$	$I \ldots 5^\circ\ 49'\ 48''.4$	$\Omega' + N' \ldots 284\ 0\ 50.3$

$\log \sin I \ldots 9.006804$ $\log \cos I \ldots 9.997747.$

DISTURBED PLANET . . . VESTA.

1836, 0h. P. M. T.	July 3.	August 14.	September 25.	November 6.	December 18.
M	324° 55'.3	336° 19'.8	347° 44'.2	359° 8'.7	10° 33'.2
E	321 48.4	334 8.0	346 34.0	359 3.8	11 33.8
log sin E	9.7913_n	9.6397_n	9.3661_n	8.2134	9.3020
e sin E	—3 6.9	—2 11.8	—1 10.2	—0 4.9	+1 0.6
$\frac{1}{2} E$	160 54.2	167 4.0	173 17.0	179 31.9	5 46.9
log sin $\frac{1}{2} E$	9.5147	9.3499	9.0680	7.9124	9.0032
log cos $\frac{1}{2} E$	9.9754_n	9.9888_n	9.9970_n	0.0000_n	9.9978
log sin $\frac{1}{2} r \sqrt{r}$	9.7196	9.5548	9.2729	8.1173	9.2081
log cos $\frac{1}{2} v \sqrt{r}$	0.1420_n	0.1554_n	0.1636_n	0.1666_n	0.1644
$\frac{1}{2} v$	159 17.4	165 55.1	172 40.3	179 29.3	6 18.6
cos $\frac{1}{2} v$	9.9710_n	9.9867_{5n}	9.9964_n	0.0000_n	9.9973
log $\sqrt{r}$	0.1710	0.1686_5	0.1672	0.1666	0.1671
v	318 34.8	331 50.2	345 20.6	358 58.6	12 37.2
log r	0.3420	0.3373	0.3344	0.3332	0.3342
u	105 43.7	118 59.1	132 29.5	146 7.5	159 46.1
log cos u	9.4331_n	9.6854_n	9.8296_n	9.9192_n	9.9723_n
log sin u	9.9835	9.9419	9.8677	9.7462	9.5388
log r sin u	0.3255	0.2792	0.2021	0.0794	9.8730
log sin v	9.8206_n	9.6740_n	9.4032_n	8.2519_n	9.3394
log cos v	9.8750	9.9453	9.9856	9.9999	9.9894
log cos E	9.8953	9.9542	9.9879	9.9999	9.9911
	0.2911	0.2966	0.2999	0.3010	0.3002
log (cos v + cos E)	0.1864	0.2508	0.2878	0.3009	0.2913
log $\frac{p}{r}$	0.0278	0.0325	0.0354	0.0366	0.0356
log $\left(\frac{p}{r} + 1\right)$	0.3152	0.3176	0.3191	0.3197	0.3192
log r sin v	0.1626_n	0.0113_n	9.7376_n	8.5851_n	9.6736
log $(p + r)$ sin v	0.4778_n	0.3289_n	0.0567_n	8.9048_n	9.9928
—p tan $\frac{1}{2} \varphi$ cos v	—0.0774	—0.0910	—0.0998	—0.1032	—0.1007
—2 r cos φ	—4.3780	—4.3310	—4.3020	—4.2900	—4.3000
log Coeff. $d L$	0.6489_n	0.6456_n	0.6437_n	0.6428_n	0.6436_n

DISTURBING PLANET . . . JUPITER.

1836, 0h. P. M. T.	July 3.	August 14.	September 25.	November 6.	December 18.
u'	191° 36′.0	195° 0′.4	198° 23′.7	201° 45′.9	205° 7′.0
log sin u'	9.3034_n	9.4132_n	9.4991_n	9.5691_n	9.6278_n
log tan u'	9.3123	9.4283	9.5219	9.6013	9.6710
λ'	191 32.3	194 55.9	198 18.3	201 39.7	205 0.0
$v + \omega'$	284 41.5	297 56.9	311 27.3	325 5.3	338 43.9
$\lambda' - (v + \omega')$	266 50.8	256 59.0	246 51.0	236 34.4	226 16.1
log sin β'	8.3102_n	8.4200_n	8.5059_n	8.5759_n	8.6346_n
log cos $[\lambda' - (v + \omega')]$	8.7405_n	9.3526_n	9.5945_n	9.7411_n	9.8396_n
log cos β'	9.9999	9.9999	9.9998	9.9997	9.9996
log sin $[\lambda' - (v + \omega')]$	9.9994_n	9.9887_n	9.9636_n	9.9215_n	9.8589_n
log $\frac{y'}{r'}$	9.9993_n	9.9886_n	9.9634_n	9.9212_n	9.8585_n
log cos γ'	8.7404_n	9.3525_n	9.5943_n	9.7408_n	9.8392_n
log sin γ'	9.9994	9.9887	9.9636	9.9216	9.8593
log r	0.3420	0.3373	0.3344	0.3332	0.3342
log x'	9.4609_n	0.0742_n	0.3172_n	0.4649_n	0.5644_n
	0.0537	0.1891	0.2926	0.2402	0.2010
log ϱ cos l'	0.3957	0.5264	0.6270	0.7051	0.7654
log ϱ sin l'	0.7199	0.7104	0.6865	0.6457	0.5845
log cos l'	9.9560	9.9226	9.8772	9.8772	9.9216
log ϱ	0.7639	0.7878	0.8093	0.8279	0.8438
log $\frac{1}{\varrho^3}$	7.7083	7.6366	7.5721	7.5163	7.4686
log $\left(-\frac{1}{r'^3}\right)$	7.8384_n	7.8348_n	7.8312_n	7.8277_n	7.8243_n
	0.5870	0.4360	0.3475	0.2910	0.2525
log Δ	7.2514_n	7.3988_n	7.4837_n	7.5367_n	7.5718_n
log y'	0.7198_n	0.7103_n	0.6863_n	0.6453_n	0.5837_n
log (1) Δ	9.2169_n	9.3643_n	9.4492_n	9.5022_n	9.5373_n
log z'	9.0307_n	9.1417_n	9.2288_n	9.3000_n	9.3598_n
log $\Delta x'$	6.7123	7.4730	7.8009	8.0016	8.1362
log $\left(-\frac{r}{\varrho^3}\right)$	8.0503_n	7.9739_n	7.9065_n	7.8495_n	7.8028_n
	0.0204	0.1647	0.6656	0.5294	0.2709
log R'	8.0299_n	7.8092_n	7.2409_n	7.4722	7.8653

FORCES AND VARIATIONS OF THE ELEMENTS.

1836 0h. P. M. T.	July 3.	August 14.	September 25.	November 6.	December 18.
$\log R_0$	9.9954_n	9.7747_n	9.2064_n	9.4377	9.8308
$\log S_0$ $\frac{42\,k}{\sqrt{p}}$	9.9367	0.0746	0.1355	0.1475	0.1210
$\log W_0$	8.2476	8.5060	8.6780	8.8022	8.8971
$d\,i$ W	9.7751_n	0.0227_n	0.1640_n	0.2524_n	0.3065_n
$d\,\Omega$ W	1.2312	1.1849	1.1078	0.9851	0.7787
$d\,\varphi$ R	0.1921_n	0.0455_n	9.7747_n	8.6234_n	9.7109
S	0.5579	0.6223	0.6593	0.6724	0.6628
$d\,\pi$ R	1.3008_n	1.3711_n	1.4114_n	1.4257_n	1.4152_n
S	1.5338_n	1.3849_n	1.1127_n	9.9608_n	1.0488
W	9.1205	9.0742	8.9971	8.8744	8.6680
$d\,\mu$ R	8.9139	8.7673	8.4965	7.3452	8.4327_n
S	0.1772_n	0.1819_n	0.1848_n	0.1860_n	0.1850_n
$d\,L$ R	0.6489_n	0.6456_n	0.6437_n	0.6428_n	0.6436_n
S	9.1216_n	8.9727	8.7005_n	7.5486_n	8.6366
W	9.1205	9.0742	8.9971	8.8744	8.6680
$42\,d\,i$	$-0''.011$	$-0''.034$	$-0''.070$	$-0''.113$	$-0''.160$
$42\,d\,\Omega$	$+0.301$	$+0.491$	$+0.611$	$+0.613$	$+0.474$
	$+1.540$	$+0.661$	$+0.096$	-0.012	$+0.348$
	$+3.123$	$+4.976$	$+6.234$	$+6.606$	$+6.079$
$42\,d\,\varphi$	$+4.663$	$+5.637$	$+6.330$	$+6.594$	$+6.427$
	$+19.777$	$+13.990$	$+4.148$	-7.302	-17.620
	-29.547	-28.807	-17.708	-1.283	$+14.783$
	$+0.002$	$+0.004$	$+0.005$	$+0.005$	$+0.004$
$42\,d\,\pi$	-9.768	-14.813	-13.555	-8.580	-2.833
	-0.0811	-0.0348	-0.0050	$+0.0006$	-0.0183
	-1.3000	-1.8050	-2.0910	-2.1552	-2.0229
$(42)^2\,d\,\mu$	-1.3811	-1.8398	-2.0960	-2.1546	-2.0412
	$+4.409$	$+2.632$	$+0.708$	-1.204	-2.981
	-0.114	-0.111	-0.069	-0.005	$+0.057$
	$+0.002$	$+0.004$	$+0.005$	$+0.005$	$+0.004$
$42\,d\,L$	$+4.297$	$+2.525$	$+0.644$	-1.204	-2.920

FOR THE INTEGRATION.

$i = 7° \; 8' \; 11''.64$ $\Omega = 103° \; 8' \; 20''.48$

1836, 0h. P. M. T.	$t - 1810.$	$42\, d\, i$	$\Delta\, i$	$42\, d\, \Omega$	$\Delta\, \Omega$
May 22	9639	000''.0	+4''.083	+0''.102	—558''.916
July 3	9681	—0.011	+4.072	+0.301	—558.615
August 14	9723	—0.034	+4.038	+0.491	—558.124
September 25	9765	—0.070	+3.968	+0.611	—557.513
November 6	9807	—0.113	+3.855	+0.613	—556.900
December 18	9849	—0.160		+0.474	

$\varphi = 5° \; 9' \; 39''.17$ $\pi = 249° \; 48' \; 26''.91$

1836, 0h. P. M. T.	$t - 1810.$	$42\, d\, \varphi$	$\Delta\, \varphi$	$42\, d\, \pi$	$\Delta\, \pi$
May 22	9639	+3''.624	—425''.567	+ 3''.886	+1170''.016
July 3	9681	+4.663	—420.904	— 9.768	+1160.248
August 14	9723	+5.637	—415.267	—14.813	+1145.435
September 25	9765	+6.330	—408.937	—13.555	+1131.880
November 6	9807	+6.594	—402.343	— 8.580	+1123.300
December 18	9849	+6.427		— 2.833	

$\mu = 978''.29671$ $L = 105° \; 53' \; 15''.63$

1836, 0h. P. M. T.	$t - 1810.$	$(42)^2\, d\, \mu$	$42\, \Delta\, \mu$	$\Delta\, M$	$42\, d\, L$	$\Delta\, L$
May 22	9639	—0''.7115	—19''.5017	—4802''.3339	+5''.825	—1071''.572
July 3	9681	—1.3811	—20.8828	—4821.8356	+4.297	—1067.275
August 14	9723	—1.8398	—22.7226	—4842.7184	+2.525	—1064.750
September 25	9765	—2.0960	—24.8186	—4865.4410	+0.644	—1064.106
November 6	9807	—2.1546	—26.9732	—4890.2596	—1.204	—1065.310
December 18	9849	—2.0412		—4917.2328	—2.920	

The constant elements belong to January 0, 1810, Paris mean time. The longitudes are referred to the mean equinox of the same epoch.

[The Method of Integration, &c. will be given in the next Number.]

Mathematical Monthly Notices.

The Mathematical Correspondent, containing new Elucidations, Discoveries, and Improvements in various branches of the Mathematics, with collections of Mathematical questions resolved by ingenious Correspondents, adapted to the present state of learning in America, and designed to inspire youth with the love of Mathematical knowledge by alluring their attention to the solutions of pleasant and curious questions, and to promote the cultivation of the Mathematics by opening a channel for the ready conveyance of discoveries and improvements from one mathematician to another. "In the mathematical sciences truth appears most conspicuous and shines in its greatest lustre." — EMERSON. Edited by GEORGE BARON. In quarterly numbers of one sheet each. Vol. I. pp. 248. New York. Printed for the Editor by Sage and Clough. 1804. This volume is in the Astor Library, New York.

The Analyst; or, Mathematical Museum, containing new Elucidations, Discoveries, and Improvements in various branches of the Mathematics, with collections of questions proposed and resolved by ingenious Correspondents. Conducted by R. ADRAIN, A. M., Fellow of the American Academy of Arts and Sciences, of the American Philosophical Society, of the Literary and Philosophical Society of New York, and Professor of Mathematics and Natural Philosophy in Columbia College, New York. Nos I. – V. pp. 150. Philadelphia: Published by William P. Farrand. 1808. No. I. New York: Printed and Published by George Long, 71 Pearl Street. 1814.

The Scientific Journal. Conducted by MR. MARRAT. Published by J. T. Murden & Co. Perth Amboy, N. J. It appears, from a volume of this work in the Astor Library, that it was published in numbers in 1818 and 1819. It contains nine numbers, February to September, 1819, and July and October, 1819. pp. 184. Either this is an imperfect volume, or the numbers were not regularly published.

The Philosophic Magazine, or Gentleman's Diary. Edited by MR. NASH. We have failed to find more than this simple announcement.

The Mathematical Diary, containing new researches and improvements in the Mathematics, with collections of questions proposed and resolved by ingenious Correspondents. In quarterly numbers. Conducted by R. ADRAIN, LL. D., F. A. P. S., F. A. A. S., &c., and Professor of Mathematics and Natural Philosophy in Columbia College, New York. Nos. I. – VIII. Vol. I. pp. 316. 1825. Nos. I. – II. Vol. II. pp. 108. 1828.

The Mathematical Miscellany. Conducted by C. GILL, Professor of Mathematics in the Institute at Flushing, Long Island. Published at the Institute. New York: W. E. Dean, Printer, No. 2 Ann Street. 1836.

This Periodical was published semiannually. The first Number appeared in February, 1836, and the eighth and last on November 1, 1839. Nos. I. – VI. constitute Vol. I. of 414 pages. Nos. VII. and VIII. contain 142 additional pages.

The Cambridge Miscellany of Mathematics, Physics, and Astronomy. To be continued quarterly. Edited by BENJAMIN PEIRCE, A. A. S., Perkins Professor of Astronomy and Mathematics in Harvard University; and JOSEPH LOVERING, A. A. S., Hollis Professor of Mathematics and Natural Philosophy in Harvard University. Nos. I. – IV., from July, 1842, to January, 1843. pp. 192. 8vo.

The above are all the facts we have been able to gather in regard to the publication of these mathematical and scientific serials; and we publish them in this incomplete state in hopes that some of our correspondents may be induced to revise them, and give the readers of the Monthly a correct account of these early and interesting periodicals.

Editorial Items.

The following gentlemen have sent us solutions of the Prize Problems in the December number of the Monthly: —

M. K. Bosworth, Sophomore Class, Marietta College, Ohio, Probs. III., IV., and V.
William Hinchcliffe, Barre Plains, Mass., Probs. I., II., III., IV., and V.; but III. and IV. came too late for the Committee.
David Trowbridge, Perry City, N. Y., Probs. III., IV., and V.
J. Q. Hollister, Hamilton College, Clinton, N. Y., Probs. III., IV., and V.
J. A. Winebrener, Princeton College, N. Y., Probs. I., II., III., IV., and V.
William Minto, University of Michigan, Probs. III., IV., and V.
C. A. Buckingham, Hamilton College, N. Y., Probs. III., IV., and V.
H. Tieman, Baltimore, Md., Probs. I., II., III., IV., and V.
Corydon C. Olney, Nunda Literary Inst., N. Y., Probs. I. and II.
Arthur M. Cazimajou, Polytechnic College, Philadelphia, Prob. V.
S. E. Benjamin, Patten, Me., Prob. III.
James F. Roberson, Senior Class, Indiana University, Prob. V.
Asher B. Evans, Madison University, Hamilton, N. Y., Probs. III., IV., and V.
F. E. Tower, Senior Class, Amherst College, Probs. III., IV., and V.
Warren Phelps, Cortlandville, N. Y., Prob. V.
William Reynolds, University of Maryland, Baltimore, Probs. I., II., III., IV., and V.
Frank N. Devereux, Boston, Mass., Probs. I. and II.
Gustavus Frankenstein, Springfield, Ohio, Probs. III. and V.

Books Received. — Report on Weights and Measures, read before the Pharmaceutical Association at their Eighth Annual Session, held in Boston, September 15, 1859. By Alfred B. Taylor, of Philadelphia, Chairman of the Committee on Weights and Measures. Boston: Press of George C. Rand & Avery, No. 3 Cornhill. 1859. . . . . . *Physical Optics.* Part II. The Corpuscular Theory of Light discussed Mathematically. By Richard Potter, A. M., F. C. P. S., &c. Cambridge: Deighton, Bell, & Co. London: Bell & Dalby. 1859. . . . . . *Of Motion.* An Elementary Treatise. By John Robert Lunn, A. M., Fellow and Lady Sadlier's Lecturer of St. John's College. Cambridge: Deighton, Bell, & Co. London: Bell & Dalby. 1859. . . . . . Address before the American Association for the Advancement of Science, August, 1859. By Professor Alexis Caswell. Published for the Association by Joseph Lovering, Permanent Secretary. 1859. . . . . . Annual Report of the Board of Regents of the Smithsonian Institution, showing its operations, expenditures, and condition for the year 1858. Washington: James B. Stedman, Printer. 1859. . . . . . The Surveyor's Companion, containing a Treatise on Mathematical Instruments, &c., &c. By William Schmolz, Mathematical-Instrument Maker, San Francisco, California. 1859. A new set of Practical Tables, useful in Surveying and Engineering, &c.; together with an improved method of Tabling, which facilitates the computation of areas and the projection of Maps. By R. C. Mathewson, U. S. Deputy Surveyor. Published by William Schmolz, San Francisco. 1859. . . . . . An Elementary Treatise on Hydrostatics, for the use of Junior University Students. By Richard Potter, A. M., F. C. P. S., late Fellow of Queen's College, Cambridge, Licentiate of the Royal College of Physicians, London, Honorary Member of the Literary and Philosophical Society of St. Andrews; Professor of Natural Philosophy and Astronomy in University College, London. Cambridge: Deighton, Bell, & Co. London: Bell & Dalby. 1859.

THE

MATHEMATICAL MONTHLY.

Vol. II. . . . APRIL, 1860. . . . No. VII.

PRIZE PROBLEMS FOR STUDENTS.

I. PROVE that an arithmetic mean is greater than a geometric.

II. Let three bodies with velocities V, V', V'', move uniformly in the same direction, in the circumference of a circle. Required the time of their conjunction, supposing them to quit a given point at the same time.

III. The diameter of a circle inscribed in the quadrant of a second circle is equal to the side of the regular octagon circumscribed about the second circle. — Communicated by Prof. HENRY H. WHITE.

IV. Required the locus of the centres of the circles inscribed within all the right-angled triangles which can be inscribed in a given semicircle. — Communicated by Prof. HENRY H. WHITE.

V. From a box containing a very large number of white and black balls, of each an equal number, three balls are taken at random and placed in a bag without being seen. A takes a ball at random from the bag, observes its color, and replaces it four times in succession. The ball was white on each of the four drawings. What are the respective probabilities that the bag contains 1, 2, or 3 white balls?

Solutions of these problems must be received by June 1, 1860.

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. IV., Vol. II.

THE first Prize is awarded to W. F. OSBORNE, of the Wesleyan University, Middletown, Ct.

The second Prize is awarded to M. K. BOSWORTH, of the Sophomore class, Marietta College, Ohio.

The third Prize is awarded to E. S. STEARNS, of Chester Institute, Chester, Morris Co., N. J.

PRIZE SOLUTION OF PROBLEM II.

By E. S. STEARNS, Chester Institute, N. J.

Show that $\dfrac{2+\sqrt{3}}{\sqrt{2}+\sqrt{(2+\sqrt{3})}}+\dfrac{2-\sqrt{3}}{\sqrt{2}-\sqrt{(2-\sqrt{3})}}=\sqrt{2}$.

We have by the formula

$$\sqrt{a\pm\sqrt{b}}=\sqrt{\frac{a+c}{2}}\pm\sqrt{\frac{a-c}{2}},$$

in which $c=\sqrt{a^2-b}$ (see algebra),

$$\sqrt{2+\sqrt{3}}=\frac{\sqrt{3}+1}{\sqrt{2}}, \text{ and } -\sqrt{2-\sqrt{3}}=\frac{1-\sqrt{3}}{\sqrt{2}}.$$

$$\therefore\quad \sqrt{2}+\sqrt{2+\sqrt{3}}=\frac{3+\sqrt{3}}{\sqrt{2}} \text{ and } \sqrt{2}-\sqrt{2-\sqrt{3}}=\frac{3-\sqrt{3}}{\sqrt{2}}.$$

Therefore the given fractions become

$$\frac{(2+\sqrt{3})\sqrt{2}}{3+\sqrt{3}}+\frac{(2-\sqrt{3})\sqrt{2}}{3-\sqrt{3}}=\frac{3\sqrt{2}+\sqrt{6}}{6}+\frac{3\sqrt{2}-\sqrt{6}}{6}=\sqrt{2}.$$

PRIZE SOLUTION OF PROBLEM III.

By W. F. OSBORNE, Wesleyan University, Middletown, Ct.

Find the roots of the equation $x^3-6x=4$, by trigonometry.

By trigonometry, $\sin 3\theta=3\sin\theta-4\sin^3\theta$; whence

(1) $$\sin^3\theta-\tfrac{3}{4}\sin\theta+\tfrac{1}{4}\sin 3\theta=0.$$

Substituting in the given equation $x=\varrho\sin\theta$, it becomes

(2) $$\sin^3\theta - \frac{6}{\varrho^2}\sin\theta - \frac{4}{\varrho^3} = 0;$$

in which any arbitrary value my be assigned to ϱ. Hence, if we put $\frac{6}{\varrho^2} = \frac{3}{4}$, or $\varrho = \sqrt{8}$, by comparing (1) and (2) we shall have

$$\sin 3\,\theta = -\frac{16}{\varrho^3} = -\frac{16}{8\sqrt{8}} = -\sqrt{\tfrac{1}{2}}. \qquad \therefore 3\,\theta = \left(2n + \frac{2 \pm 3}{4}\right)\pi.$$

Hence, $\sin\theta = \sin\left(\frac{2n}{3} + \frac{2 \pm 3}{12}\right)\pi$, from which all the different values of $\sin\theta$ may be obtained by making $n = 0$ or ± 1. Therefore the three values of x are

$$-\sqrt{8}\sin 15^\circ = 1 - \sqrt{3},\quad \sqrt{8}\cos 15^\circ = 1 + \sqrt{3},\quad -\sqrt{8}\cos 45^\circ = -2.$$

Prize Solution of Problem IV.

By W. F. Osborne, Wesleyan University, Middletown, Ct.

Four persons, A, B, C, D, in order, beginning with A, cut a pack of cards, replacing them after each cut, on condition that the first who cuts a heart shall win. What are their respective probabilities of success?

The probability that a heart will be cut on the first trial is evidently $\frac{1}{4}$, and $\frac{3}{4}$ against it; on the second trial $\frac{1}{4} \times \frac{3}{4}$ for, and $\frac{3}{4} \times \frac{3}{4}$ against it; on the third trial $\frac{1}{4} \times \frac{3}{4} \times \frac{3}{4}$ for, and $\frac{3}{4} \times \frac{3}{4} \times \frac{3}{4}$ against it; and so on to the nth trial, on which the probability that a heart will be cut is $\frac{1}{4}\left(\frac{3}{4}\right)^{n-1}$, and $\left(\frac{3}{4}\right)^n$ against it.

But A has the first, fifth, ninth, &c. cuts; hence his probability of success is

$$\tfrac{1}{4}\left(1 + \left(\tfrac{3}{4}\right)^4 + \left(\tfrac{3}{4}\right)^8 + \text{\&c.}\right) = \tfrac{1}{4} \cdot \frac{1}{1 - \left(\frac{3}{4}\right)^4} = \tfrac{64}{175}.$$

B has the second, sixth, &c. cuts, and hence his probability of success is

$$\tfrac{1}{4} \cdot \tfrac{3}{4}\left(1 + \left(\tfrac{3}{4}\right)^4 + \left(\tfrac{3}{4}\right)^8 + \text{\&c.}\right) = \tfrac{1}{4} \cdot \tfrac{3}{4} \cdot \frac{1}{1 - \left(\frac{3}{4}\right)^4} = \tfrac{48}{175}.$$

In precisely the same manner C's and D's probabilities of success are

$$\tfrac{1}{4} \cdot \left(\tfrac{3}{4}\right)^2 \cdot \frac{1}{1 - \left(\frac{3}{4}\right)^4} = \tfrac{36}{175}, \qquad \tfrac{1}{4} \cdot \left(\tfrac{3}{4}\right)^3 \cdot \frac{1}{1 - \left(\frac{3}{4}\right)^4} = \tfrac{27}{175}.$$

Prize Solution of Problem V.

By Asher B. Evans, Madison University, Hamilton, N. Y.

The notation of Problem V. in the November number of the Monthly being retained, prove that in the plane

(1) $$\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C = \cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C;$$

and in the sphere

(2) $$\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C = \cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C\,\frac{\cos(\rho+\delta)\cos(\rho-\delta)}{\cos^2 r\cos^2\rho},$$

(3) $$\cot\tfrac{1}{2}A\,\frac{\cos\delta'}{\cos\rho'} + \cot\tfrac{1}{2}B\,\frac{\cos\delta''}{\cos\rho''} + \cot\tfrac{1}{2}C\,\frac{\cos\delta'''}{\cos\rho'''} = \cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C\,\frac{\cos\delta}{\cos\rho}.$$

In the plane we have

$$\cot\tfrac{1}{2}A = \frac{s(s-a)}{n},\quad \cot\tfrac{1}{2}B = \frac{s(s-b)}{n},\quad \cot\tfrac{1}{2}C = \frac{s(s-c)}{n},$$

in which $s = \frac{1}{2}(a+b+c)$, $n^2 = s(s-a)(s-b)(s-c)$.

$$\therefore\ \cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C = \frac{s^2}{n};$$

$$\therefore\ \cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C = \frac{s^2(s-a)(s-b)(s-c)}{n^3} = \frac{s^2}{n};$$

therefore (1) is true. In the sphere

$$\frac{\cot\frac{1}{2}A}{\sin s} = \frac{\sin(s-a)}{n},\quad \frac{\cot\frac{1}{2}B}{\sin s} = \frac{\sin(s-b)}{n},\quad \frac{\cot\frac{1}{2}C}{\sin s} = \frac{\sin(s-c)}{n},$$

in which

$$s = \tfrac{1}{2}(a+b+c),\quad n^2 = \sin s\sin(s-a)\sin(s-b)\sin(s-c).$$

(4) $$\therefore\ \cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C = \frac{\sin s\sin(s-a)\sin(s-b)\sin(s-c)}{n};$$

(5) $$\therefore\ \cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C = \frac{\sin^2 s}{n}.$$

Also

$$\frac{\cos(\varrho+\delta)\cos(\varrho-\delta)}{\cos^2 r\cos^2\varrho} = \frac{\cos^2\delta - \sin^2\varrho}{\cos^2 r\cos^2\varrho} = \tan^2\varrho\left(\frac{\cos^2\delta}{\cos^2 r\sin^2\varrho} - 1 + \tan^2 r\right).$$

But we have (Chauvenet's Trig. Eqs. 307, 319, 332)

$$\tan r = \frac{2\sin\frac{1}{2}a\sin\frac{1}{2}b\sin\frac{1}{2}c}{n},\quad \tan\varrho = \frac{n}{\sin s},$$

$$\frac{\cos^2\delta}{\cos^2 r\sin^2\varrho} - 1 = \left(\frac{\sin s + 2\sin\tfrac{1}{2}a\sin\tfrac{1}{2}b\sin\tfrac{1}{2}c}{n}\right)^2.$$

$$\therefore\ \frac{\cos(\varrho+\delta)\cos(\varrho-\delta)}{\cos^2 r\cos^2\varrho} = \frac{\sin s + 4\sin\tfrac{1}{2}a\sin\tfrac{1}{2}b\sin\tfrac{1}{2}c}{\sin s}$$

(6) $$= \frac{\sin(s-a)+\sin(s-b)+\sin(s-c)}{\sin s}.$$

Combining (4), (5), (6), we get (2).

Again, since (Math. Monthly, Vol. II. p. 150)

$$\frac{\cos\delta'}{\cos\varrho'} = \cos r\left(\frac{-\sin a+\sin b+\sin c}{2\sin(s-a)}\right),$$

we have

$$\cot\tfrac{1}{2}A\,\frac{\cos\delta'}{\cos\varrho'} = \frac{\sin s(-\sin a+\sin b+\sin c)\cos r}{2n},$$

$$\cot\tfrac{1}{2}B\,\frac{\cos\delta''}{\cos\varrho''} = \frac{\sin s(\sin a-\sin b+\sin c)\cos r}{2n},$$

$$\cot\tfrac{1}{2}C\,\frac{\cos\delta'''}{\cos\varrho'''} = \frac{\sin s(\sin a+\sin b-\sin c)\cos r}{2n}.$$

(7) $$\therefore\ \cot\tfrac{1}{2}A\,\frac{\cos\delta'}{\cos\varrho'} + \cot\tfrac{1}{2}B\,\frac{\cos\delta''}{\cos\varrho''} + \cot\tfrac{1}{2}C\,\frac{\cos\delta'''}{\cos\varrho'''} = \frac{\sin s(\sin a+\sin b+\sin c)\cos r}{2n}.$$

But (CHAUVENET'S Trig. Eq. 332)

(8) $$\frac{\cos\delta}{\cos r\sin\varrho} = \frac{\sin a+\sin b+\sin c}{2n},\quad \cos r\tan\varrho = \frac{n\cos r}{\sin s}.$$

Combining equations (8) we get

(9) $$\frac{\cos\delta}{\cos\varrho} = \frac{(\sin a+\sin b+\sin c)\cos r}{2\sin s}.$$

Combining (5) and (7)

(10) $$\cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C\,\frac{\cos\delta}{\cos\varrho} = \frac{\sin s(\sin a+\sin b+\sin c)\cos r}{2n},$$

and combining (7) and (10) we have (3)

Solution of Part First.

By Gustavus Frankenstein, Springfield, Ohio.

The plane triangle $C = 180 - (A + B)$,

$$\therefore \cot \tfrac{1}{2} C = \cot \left(90 - \tfrac{1}{2}(A + B)\right) = \tan \tfrac{1}{2}(A + B)$$

$$= \frac{1}{\cot \frac{1}{2}(A + B)} = \frac{\cot \frac{1}{2} A + \cot \frac{1}{2} B}{\cot \frac{1}{2} A \cot \frac{1}{2} B - 1},$$

$$\therefore \cot \tfrac{1}{2} C (\cot \tfrac{1}{2} A \cot \tfrac{1}{2} B - 1) = \cot \tfrac{1}{2} A + \cot \tfrac{1}{2} B,$$

$$\therefore \cot \tfrac{1}{2} A \cot \tfrac{1}{2} B \cot \tfrac{1}{2} C = \cot \tfrac{1}{2} A + \cot \tfrac{1}{2} B + \cot \tfrac{1}{2} C.$$

Simon Newcomb.
W. P. G. Bartlett.
Truman Henry Safford.

NOTES AND QUERIES.

1. *On some properties of the powers of the same number.*—The author announced the discovery of some general laws which regulate the series of the powers of any number. For instance, in the following series of the powers of 5, the number of digits in the several recurrent vertical series may be expressed by the powers of 2.

No. of digits recurring.	Powers of 5.		The vertical series are
	5	1st	5
1	25	2d	2
2	125	3d	16
	625	4th	3580
4	3125	5th	17956240
8	15625	6th	3978175584236200
	78125	7th	19840377976181556439582242163600
16 . . .	390625	8th	
32 . .	1953125	9th	
	9765625	10th	
64 . .	48828125	11th	
128 .	244140625	12th	
256 .	1220703125	13th	

The next consists of 64 figures, and so on. He pointed out that a similar law existed for every other number, and he exhibited formulæ by which the sum of any of the recurrent series may be de-

termined. In the case of 5, $S_n = 2(S_{n-1} + 1)$, the consecutive sums of the several series being 7, 16, 34, 70, 142, &c. In this way tables of the powers of numbers may be constructed to any extent whatever with very little labor. This discovery will enable certain calculations to be made with a degree of accuracy hitherto impossible. — J. Pope Hennessy, of the Inner Temple, Report of the British Assoc. for the Adv. of Science. Leeds. 1858.

2. The *celestial* part of the problem on page 204 of the March number of the Monthly is very simply solved in a general form. Employing the usual notation, S the integral with respect to the mass, Ω the *potential*, $\triangle$ the mutual distance of m and m', &c., *the law of power* of any system of *fixed* forces is

$$(1) \qquad \tfrac{1}{2} S(m v^2) = \Omega + H.$$

$$(2) \qquad \Omega = \mu\, S\, S[m\, m'\, \varphi(\triangle)]. \qquad (3) \quad \varphi(\infty) = 0.$$

Fixed forces might, it is true, be imagined which would not satisfy (2) and (3); but it can hardly be supposed that such would ever have been introduced into a system of nature.

The problem here considered is to find the form of the function φ, under the condition that two systems, whose linear dimensions are in the ratio n, shall present similar appearances at the same times. If (1) expresses the law of power of one system, that of the other is by this condition,

$$\tfrac{1}{2} S[n^3 m (n v)^2] = \mu\, S\, S[n^6 m\, m'\, \varphi(n \triangle)] + H_n.$$

Whence

$$S\, S\, m\, m' [n\, \varphi(n \triangle) - \varphi(\triangle)] = 0 \ldots$$

$$(4) \qquad \therefore\ n\, \varphi(n \triangle) - \varphi(\triangle) = 0.$$

The derivative of (4) with reference to n is

$$(5) \qquad \varphi(n \triangle) + n \triangle\, \varphi'(n \triangle) = 0.$$

The integral of (5) is $\varphi(n\triangle) = \frac{a}{n\triangle}$. The *only* form of potential which can satisfy the present problem is therefore that which results in *the law of gravity.* — B.

3. Reply to query on page 186, March No. For the integral of

$$D_r^2 \varphi(r) + \frac{D_r \varphi(r)}{r} \pm m^2 \varphi(r) = 0,$$

see De Morgan's *Calculus*, pp. 699, 700; Boole's *Differential Equations*, pp. 455, 459; or Carmichael's *Calculus of Operations*, p. 54. — B.

NOTE ON EQUATIONS OF THE SECOND DEGREE.

By Dr. R. C. Matthewson, U. S. Deputy Surveyor, San Francisco, Cal.

Every equation of the second degree, involving one unknown quantity, may be reduced to one of the following four forms, in which the coefficient of the second power of the unknown quantity will always be unity: —

1st, $x^2 - 2rx = a^2$; 2d, $x^2 + 2rx = a^2$;
3d, $x^2 - 2rx = -a^2$; and 4th, $x^2 + 2rx = -a^2$.

The roots of the 2d form differ from those of the 1st, and the roots of the 4th from those of the 3d, only in the change of signs; because if the signs of the alternate terms of an equation be changed, the signs of all the roots will be changed, while their magnitudes remain unchanged.

It is obvious, therefore, that the four forms may be reduced to two classes, the 1st class comprising the 1st and 2d forms, in which the absolute term is *positive*, and the 2d class comprising the 3d and 4th forms, in which the absolute term is *negative*.

In both classes, considered as lineal equations, one half of the coefficient of the first power of the unknown quantity represents

the *radius* of a circle, while the square root of the absolute term, in the 1st class, represents a *tangent,* and in the 2d class, a *sine* of that circle.

In the first class one of the roots of the equation represents the *external secant,* and the other the *diameter* of the circle *increased* by the *external secant,* while in the 2d class one of the roots represents the *versed sine* and the other the *diameter* of the circle *diminished* by the *versed sine.*

As there are no conditions in any of the four forms to indicate in what *direction* the lines are to be drawn, it follows that, in the 1st class, the external secant as well as the sum of the diameter and external secant, and in the 2d class, the versed sine as well as the difference of the diameter and versed sine, may be taken either positive or negative.

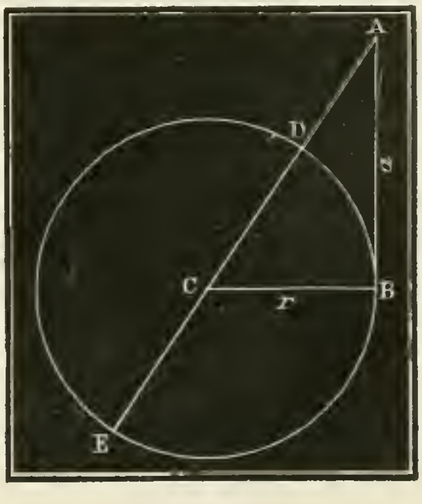
Hence, it is plain, that in each class, the two roots of one of the forms are respectively equal to the two roots of the other in magnitude, but have contrary signs, as has been already shown, and may be seen at once by comparing the roots together, as follows: —

Roots of 1st Class.

Form 1st, $x^2 - 2rx = a^2$, or $x = r \pm \sqrt{r^2 + a^2}$,

Form 2d, $x^2 + 2rx = a^2$, or $x = -r \pm \sqrt{r^2 + a^2}$.

Construction of 1st Class.



With the radius $CB = r$, half the coefficient of x, describe a circle; draw the tangent $BA = a$, the square root of the absolute term; and through A and C draw the secant AC produced, cutting the circumference of the circle in D and E; then AD and AE will be the roots of the equation, the former of which is the external secant and the latter the diameter of the circle increased by the external secant.

Demonstration of 1st Class.

Form 1st, $AD = \quad r - \sqrt{r^2 + a^2} = \quad$ rad — sec = — ext. secant,

" $\quad AE = \quad r + \sqrt{r^2 + a^2} = \quad$ rad + sec = + dia + ext. sec.,

Form 2d, $AD = -r + \sqrt{r^2 + a^2} = -$ rad + sec = + ext. secant,

" $\quad AE = -r - \sqrt{r^2 + a^2} = -$ rad — sec = — dia — ext. sec.

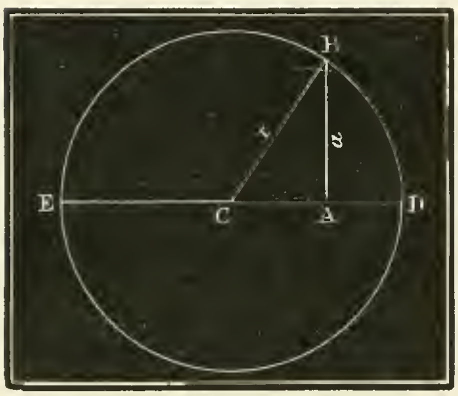
Roots of 2d Class.

Form 3d, $\quad x^2 - 2rx = -a^2, \quad$ or $x = \quad r \pm \sqrt{r^2 - a^2},$

Form 4th, $\quad x^2 + 2rx = -a^2, \quad$ or $x = -r \pm \sqrt{r^2 - a^2}.$

Construction of 2d Class.

With the radius $CD = r$, half the coefficient of x, describe a circle; through the centre C draw a diameter cutting the circumference of the circle in D and E; draw the sine $AB = a$, the square root of the absolute term, and join B and C; then AD and AE will be the roots of the equation, the former of which is the versed sine and the latter the diameter of the circle diminished by the versed sine.



Demonstration of 2d Class.

Form 3d, $AD = \quad r - \sqrt{r^2 - a^2} = \quad$ rad — cos = + versed sine,

" $\quad AE = \quad r + \sqrt{r^2 - a^2} = \quad$ rad + cos = + dia — ver. sine,

Form 4th, $AD = -r + \sqrt{r^2 - a^2} = -$ rad + cos = — versed sine,

" $\quad AE = -r - \sqrt{r^2 - a^2} = -$ rad — cos = — dia + ver. sine.

SOLUTION OF PROBLEMS IN PROBABILITIES.

By Dr. Joel E. Hendricks, Newville, Indiana.

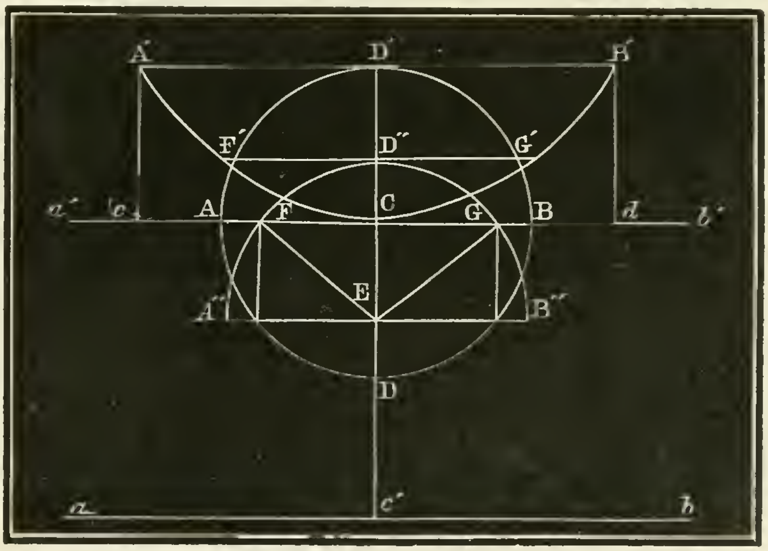
I have recently had my attention directed to a question somewhat analogous to the Prize Question proposed in the *Lady's and Gentle-*

man's Diary, for 1859, and published in the Mathematical Monthly, Vol. I. No. IV.; and have succeeded, as I think, in obtaining a solution, which, by a slight and obvious extension, will also apply to the "Prize Question." I submit the solutions for publication.

Prob. I. "A plane surface is ruled with parallel or equidistant lines; a slender rod, the length of which equals the perpendicular distance between two consecutive lines, is thrown at hazard upon the plane; determine the probability of its falling across a line."

Prob. II. "Two rods, two and four feet long respectively, having their middle points connected by a string one foot in length, are thrown up; show that their chance of crossing is $\frac{1}{2} + \frac{2}{\pi^2}$."

Solution.—Let $a\,b$, $a'\,b'$, represent two consecutive lines, and Cc' the perpendicular distance between them; also, let $CD = \frac{1}{2}\,Cc'$ = one half the length of the rod.



The centre of the rod may fall anywhere on the line Cc', but as the position of the rod in relation to the line $a\,b$ will be the same if it fall below D, as to the line $a'\,b'$ if it fall above, we need only consider its position when the centre falls between C and D.

Let E be any point between C and D, and upon E as a centre and radius $= CD$ describe the semicircle $A''\,D''\,B''$, cutting the line $a'\,b'$ in F and G. Then it is obvious, that, in all positions between F and G, the rod will cross the line, and in all positions between F and A'', and G and B'', it will miss. Therefore, the probability of the rod's falling across the line $a'\,b'$ will be represented by a fraction

whose numerator is the sum of all the arcs above the line $a' b'$ made by causing E to assume all possible positions between C and D; and whose denominator is the sum of all the semicircles $A'' D'' B''$, &c., made by causing E to assume all possible positions between C and D. But the sum of all the arcs above $a' b'$ is represented by the area of the sinusoid $A' F' C G' B'$; and the sum of all the semicircles $A'' D'' B''$, &c., is represented by the area of the rectangle $A' B' c d$, which, when the radius $CD = 1$, is π.

Let the ordinate $F' D''$ ($=$ arc $F D''$) $= y$; then is $C D'' = 1 - \cos y$, and the area of the sinusoid $A' F' C G' B'$ is

$$2\int_0^{\frac{1}{2}\pi} y \sin y \, dy = \Big[- 2 y \cos y + 2 \sin y\Big]_0^{\frac{1}{2}\pi} = 2.$$

Therefore the probability of the rods falling across a line is $\frac{2}{\pi}$.

In extending this solution to Problem II. we observe, (if $a' b'$ represent the long rod, $A'' B''$ the short one, and $C D$ the string,) that, supposing the long rod to fall in the position $a' b'$, the centre of the short rod may fall anywhere in the semicircle $A D B$. Therefore the chance of their crossing will be represented by a fraction whose numerator is the sum of all the products $F' G' \times F G$ obtained by making E assume all possible positions between C and D; and whose denominator is the sum of all the semicircles $A'' D'' B''$, &c., obtained by making E assume all possible positions in the semicircle $A D B$. Therefore the chance of crossing is

$$\frac{4}{\frac{1}{2}\pi^2}\int_0^{\frac{1}{2}\pi} y \sin^2 y \, dy = \frac{4}{\frac{1}{2}\pi^2}\int_0^{\frac{1}{2}\pi} y \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2 y\right) dy$$

$$= \left[\frac{y^2 - y \sin 2 y - \frac{1}{2}\cos 2 y}{\frac{1}{2}\pi^2}\right]_0^{\frac{1}{2}\pi}$$

$$= \frac{\frac{1}{4}\pi^2 - 0 + \frac{1}{2} + \frac{1}{2}}{\frac{1}{2}\pi^2} = \frac{1}{2} + \frac{2}{\pi^2}.$$

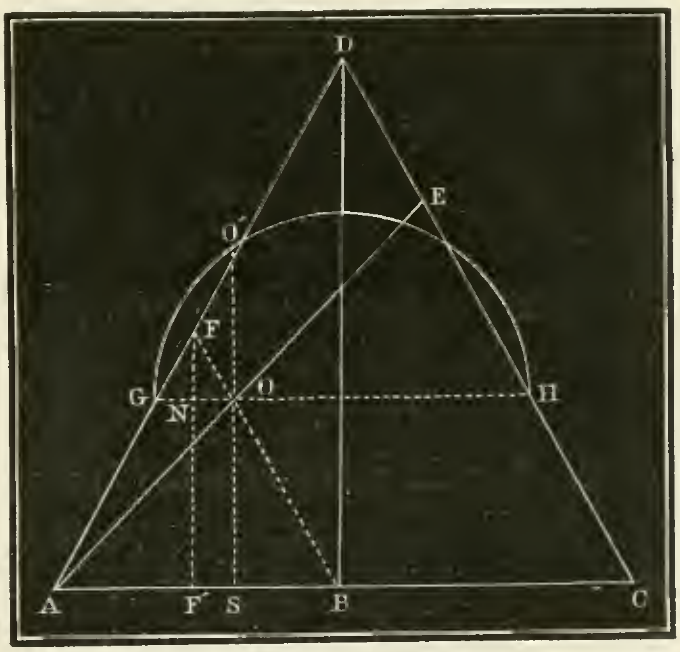
ON THE INTERPRETATION OF IMAGINARY ROOTS IN QUESTIONS OF MAXIMA AND MINIMA.

By Major BENJAMIN ALVORD, U. S. Army.

WE shall take the following example:—

To cut from a given right cone an ellipse whose surface is a maximum, or a minimum, by a plane passing through a fixed point on one of its elements.

Let D be the vertex, $A\,C$ the diameter of the base of the given right cone, and A the given fixed point. Suppose $A\,E$ to be the transverse axis of the required ellipse. Put $A\,B$, the radius of the base $= a$, and the altitude $B\,D = c$. Through B draw $B\,F$ parallel to $C\,D$. The centre, O, of the required ellipse must be found somewhere on $B\,F$. Through O draw a plane, $G\,H$, parallel to the base; it will cut out of the cone a circle, as $H\,O'\,G$, having $H\,G$ as a diameter. Erect the perpendicular $O\,O'$, which is the semi-conjugate axis of the required ellipse. The surface of the ellipse is measured by $\pi\,A\,O \times O\,O'$; or the surface is a maximum (or minimum) when the product $A\,O \times O\,O'$ is a maximum; or the square $A\,O^2 \times O\,O'^2$; or when $A\,O^2 \times O\,G \times O\,H$ is a maximum, since $O\,O'^2 = O\,G \times O\,H$. But $O\,H = B\,C = a$ is a constant. Thus the function is then expressed $u = A\,O^2 \times O\,G$.



First, we have

(1) $$O\,G : A\,B = N\,F : F\,F'.$$

But $N\,F = \frac{c}{2} - O\,S$, $F\,F' = \frac{c}{2}$; and putting $O\,G = y$ (1) becomes

$$y : a = \frac{c}{2} - O S : \frac{c}{2}.$$

$$\therefore \frac{c y}{2} = \frac{a c}{2} - a \times O S;$$

$$\therefore O S = \frac{a c - c y}{2 a} = \frac{c}{2} - \frac{c y}{2 a}.$$

But $A S = A F' + F' S = \frac{a}{2} + \frac{y}{2}$;

$$\therefore A O^2 = O S^2 + A S^2 = \frac{c^2}{4} - \frac{c^2 y}{2 a} + \frac{c^2 y^2}{4 a^2} + \frac{a^2}{4} + \frac{a y}{2} + \frac{y^2}{4}.$$

$$\therefore u = A O^2 \times O G = \frac{c^2 y}{4} - \frac{c^2 y^2}{2 a} + \frac{c^2 y^3}{4 a^2} + \frac{a^2 y}{4} + \frac{a y^2}{2} + \frac{y^3}{4}.$$

$$\frac{d u}{d x} = \frac{c^2}{4} - \frac{c^2 y}{a} + \frac{3 c^2 y^2}{4 a^2} + \frac{a^2}{4} + a y + \frac{3 y^2}{4} = 0.$$

$$\therefore (3 c^2 + 3 a^2) y^2 - (4 a c^2 - 4 a^3) y = - a^4 - a^2 c^2;$$

(2) $$\therefore y = \frac{2 a (c^2 - a^2)}{3 (a^2 + c^2)} \pm \frac{a \sqrt{c^4 + a^4 - 14 a^2 c^2}}{3 (a^2 + c^2)}.$$

Finding the second derivative, we have

$$\frac{d^2 u}{d y^2} = \frac{3}{2} \left(\frac{a^2 + c^2}{a^2}\right) y - \frac{c^2 - a^2}{a}.$$

Substituting the value of y in (2) we get

$$\frac{d^2 u}{d y^2} = \pm \frac{\sqrt{c^4 + a^4 - 14 a^2 c^2}}{2 a}.$$

Hence, when the value of y is taken with the *plus* sign of the radical, it corresponds to the required minimum; but when the *minus* sign is taken, it corresponds to a maximum.

But the authorities lay it down (as in CHURCH's Calculus, Art. 67), that only the *real* roots of the equation $\frac{d u}{d y} = 0$ answer to a maximum or a minimum. In order that the value of y, or of the second derivative, shall not become imaginary, it is necessary that $a^4 + c^4 - 14 a^2 c^2$ should be positive. But this expression can be put under the form

$$(a^2 + c^2)^2 - 16\, a^2 c^2 = (a^2 + c^2 + 4\, a\, c)\, (a^2 + c^2 - 4\, a\, c).$$

It is necessary then that $a^2 + c^2 > 4\, a\, c$. The limit of the relation between a and c will be found by putting $a^2 + c^2 = 4\, a\, c$.

(3) $\quad \therefore\; c^2 - 4\, a\, c = -\, a^2;\quad$ or $c = 2\, a + a \sqrt{3} = a\,(3.732)$.

But the angle $A\,D\,B$, answering to these relations between the radius of the base and the altitude, is 15° 0′ 1″. Thus, in order that the problem may be possible, or that a maximum or minimum may exist, the angle $A\;D\;C$, or the angle between the two elements of the cone cut by a section through the axis, must be less than 30° 0′ 2″; or the altitude of the cone must be more than 3.732 times the radius of the base. In the first figure the angle $A\;D\;C$ is 57°, and hence in that example there is no maximum or minimum.

Take the example in the annexed figure. The angle $A\;D\;C$, at the vertex is 26°. It will be found that the position of the section for a maximum is $A\,K$, and for a minimum it is $A\,I$. If, in equation (3), the minus sign of the radical were taken, we should find that $c = a\,(0.268)$. In this case, the expression under the radical would not be imaginary if the altitude is less than 0.268 of the base; or, which is the same thing, if the base is more than 3.732 times the altitude; or the angle $A\;D\;C$ more than 149° 59′ 58″. In this case, $c - a$ in equation (2) being negative, the value of y is negative for both signs of the radical. Therefore y, or $G\;O$, in the first figure is laid off in the opposite direction, and the line $A\,E$ passes to the outside of the angle $C\,A\,D$. The problem in this case would have no reference to the cone, but could be announced in the following form. Having given an isosceles triangle, $A\,D\,C$, and $B\,F$ being drawn parallel to $C\,D$, required to draw a line, as $A\,E$, through the point A, such that the product $A\;O \times O\;G$ is a maximum or a minimum.

This discussion is given as a remarkable example in illustration of the fact, that if, in the investigation of questions in maxima and minima, an imaginary expression for the unknown quantity is found, it indicates that certain relations between the constants in the original function must be fulfilled in order that the problem may be possible.

LETTER OF LE VERRIER AND REMARKS OF FAYE UPON THE INTER-MERCURIAL PLANETS.

Translated by Charles Henry Davis, Commander U. S. N., Supt. Nautical Almanac.

My dear Sir: —

I have the pleasure to communicate to you a translation of the remarkable letter of M. LeVerrier, published in the *Comptes Rendus*, No. 11, Deuxième Semestre, 1859, together with M. Faye's remarks upon this occasion.

The important bearing of these papers upon the total eclipse of next July, induces me to ask you to bring them, in the pages of your Journal, to the notice of astronomical observers throughout the country.

I am, very respectfully,

Your friend and obedient servant,

CHARLES HENRY DAVIS,

Commander U. S N., Superintendent N. A.

To J. D. Runkle, A. M., A. A. S.,

Editor of the Mathematical Monthly, Assistant N. A.

Letter of M. LeVerrier to M. Faye upon the Theory of Mercury, and upon the Motion of the Perihelion of that Planet.

You have probably not forgotten, that, in my investigations into the motions of our planetary system, I have encountered difficulties in producing a complete agreement between theory and observa-

tion. This agreement, said Bessel, thirty years ago, has always been asserted, without, however, having ever been confirmed in a manner sufficiently substantial.

The known errors in the motion of Uranus have been explained by the discovery of Neptune.

The investigation of the difficulties presented by the Sun has been long and intricate. It was necessary first to revise the catalogue of the fundamental stars, in order to exclude from it any systematic error. I have resumed, subsequently, the whole theory of the inequalities of the Earth; after which, I have been led to the consecutive discussion of as many as nine thousand observations of the Sun, made in different observatories. This work has proved, that the meridian observations have not perhaps always possessed the accuracy attributed to them; and thus the errors, described at first as belonging to theory, must in the end be thrown back on the uncertainty of the observations.

The theory of the Sun being put out of the question, the study of the motions of Mercury could be taken up with advantage. And it is concerning this undertaking that I wish to address you at present.

While we possess, in respect to the Sun, meridian observations only, liable to great objections, we arrange in order, through the space of a century and a half, a certain number of observations of Mercury, possessing great precision. I am speaking here of the internal contacts of the disc of Mercury with the disc of the Sun, when the planet passes in front of the latter. Provided the place of observation has been well known, provided the astronomer has been furnished with a decent telescope, and his clock has been regulated within some seconds, the knowledge of the moment when the internal contact took place ought to furnish the means of estimating the distance between the centres of the planet and of the

Sun, without a greater error than one second of arc. We have, from 1697 to 1848, twenty-one observations of this sort, which we ought to be able to satisfy in the strictest manner, if the inequalities in the motions of the Earth and of Mercury have been correctly computed, and if the values assigned to the disturbing masses are exact.

In my first researches on Mercury, published in 1842, the observations of the transits were not produced with such great accuracy. There might be observed, among other errors, in relation to the transits of the month of May, a somewhat remarkable progressive error, which amounted to 9 seconds of arc in 1793. Such discrepancies could not be attributed to errors of observation. But, not having then revised my theory of the Sun, I thought it best to refrain from drawing any conclusion from this fact.

The use of the corrected tables of the Sun, in my new work, has not caused the errors above described to disappear entirely; systematic errors, which could not be ascribed to the observations without supposing that astronomers, such as Lalande, Cassini, Bouguer, &c., had committed errors of several minutes of time, having even a progressive variation from one epoch to another, — an impossibility!

But it is worthy of remark, that an addition of thirty-eight seconds to the secular motion of the perihelion is sufficient to represent all the observations of the transits within less than a second, and even the greater part within less than half of a second. This neat result, which immediately gives to all the comparisons a precision superior to that hitherto arrived at in astronomical theories, shows clearly that this increase of the motion of the perihelion of Mercury is indispensable, and that with it the tables of Mercury and the Sun possess all the accuracy required.

The necessity for adding thirty-eight seconds to the secular mo-

tion of the perihelion of Mercury being once admitted, let us inquire to what conclusions it will lead us. As the motion originally adopted for the perihelion proceeded from received values of the masses of the disturbing planets, we ought first to inquire what changes it would be necessary to apply to these masses in order to increase the computed motion by thirty-eight seconds. Now we perceive that this would only be possible upon one condition, namely, to increase the mass assigned to Venus *by at least a tenth* of its value. Is this alteration admissible?

If we derive the mass of Venus from the periodic perturbations which that planet occasions to the Earth's motion, we find, from the discussion of numerous meridian observations of the Sun, made between 1750 and 1810, that this mass is the *four-hundred-thousandth* part of that of the Sun. We reach the same result by taking into account the observations made between 1810 and 1850. It is the mass which we have adopted, and which it would be necessary to increase by *a tenth,* according to the discussion of the transits of Mercury over the Sun.

The perturbative action of Venus is again perceptible in the secular variation of the obliquity of the ecliptic; and if we deduce this variation from the seven solstices observed with the greatest accuracy, from the time of Bradley to the present day, we find that the mass of Venus, which we have just quoted, is *a little too great,* — *a result contrary to that given by Mercury.* This contradiction is the point to which we are to give our attention.

If we inquire how the seven solstices, observed since Bradley, could be represented, supposing a variation of the obliquity of the ecliptic which would correspond to a mass of Venus increased by *one tenth,* we perceive that it is impossible to avoid errors of *two seconds and a half* in the measured value of the obliquity. It seems difficult to admit this result, the more especially that the errors

would vary progressively from Bradley to our own epoch, which would in fact establish a difference of five seconds between the extreme observations.

A grave difficulty then results from the comparison of the theories of the Earth and of Mercury, which appear to involve different values for the mass of Venus. If we accept the mass given by the observations of Mercury, we must conclude, either that the secular variation of the obliquity of the ecliptic, deduced from observation, contains errors that are very improbable, or else that this obliquity undergoes changes from other causes, not yet recognized by us. If, on the other hand, we consider the variation of the obliquity and the causes by which it is produced as perfectly settled, we shall be induced to think that the excess in the motion of the perihelion of Mercury is owing to some agency still unknown, "cui theoriæ lumen nondum accesserit."

It is far from my intention to decide absolutely between these hypotheses. It has been my wish only to show that there is a serious difficulty here, that deserves to arrest the attention of astronomers, to become the subject of their deep thought, and to furnish matter for a weighty discussion. In order to take the first step in this direction, I will observe that it is not apparent what disturbing cause could derange the obliquity of the ecliptic, without, at the same time, exerting very conspicuous effects upon the secular variations of the elements of motion of the planets, — effects which have not been noticed, — whilst it would be possible to conceive a cause capable of producing in the perihelion of Mercury the thirty-eight seconds of secular motion required, and which would produce in the planetary system no other sensible effect whatever.

To fix our thoughts, let us consider a planet which would be situated between Mercury and the Sun, and, as we have not observed a variation in the motion of the node of the orbit of Mercury

similar to that of the perihelion, let us conceive that the supposed planet moves in an orbit but little inclined to the orbit of Mercury. Let us even conceive, on account of the indeterminateness of the problem, that the orbit is circular.

As the hypothetical planet must produce a secular motion of thirty-eight seconds in the perihelion of Mercury, it follows that there is a relation between its mass and its distance from the Sun, such that, in proportion as we assume the distance to be smaller, the mass will increase, and inversely. For a distance a little less than half of the mean distance of Mercury from the Sun, the mass sought for will be equal to that of Mercury.

But is it possible such a body could exist without having ever been noticed? Assuredly it would be endowed with great brilliancy. Is it to be believed that in consequence of its slight elongation it would be always lost in the diffused light of the Sun? How conceive that we have never been struck with its brilliant light during some one of the total eclipses of the Sun? How does it happen that we have never discovered it passing over the disc of the Sun?

All these difficulties vanish upon the admission, in the place of a single planet, of the existence of a series of small bodies (corpuscules) circulating between Mercury and the Sun.

Under the *mechanical* aspect, the influence of all these small bodies would be united to produce the required motion of the perihelion of Mercury; and, supposing always that they moved in circles, they would exert no effect upon the eccentricity of the orbit of this planet. As they would be distributed over all parts of the ring that they would form, the periodic influences which each one would exert upon Mercury would mutually destroy each other.

Under the *physical* aspect, it would not be astonishing if the regions which border on the Sun should be found to be less free than

the remainder of the planetary system. Since there circulates between Jupiter and Mars a ring of small bodies, of which the largest alone have been seen in our telescopes; since everything leads us to the belief that the vicinity of the orbit of the Earth is furrowed by innumerable groups of asteroids, it is altogether natural to suppose that the same formation may be reproduced within the orbit of Mercury. Could any of these bodies be sufficiently noticeable to be perceived in their transits over the disc of the Sun? Astronomers, already so observant of all the phenomena which exhibit themselves upon the surface of this star, will undoubtedly find, in these reflections, an additional motive for following with attention the smallest and best defined spots. Some minutes of observation will be usefully employed in deducing their nature from observing their motions.

Here, then, my dear associate, is a new complication which shows itself in the neighborhood of the Sun; there, where M. Encke has already pointed out so important a one to us in the subject of his comet of short period. This induces me to hope that both you and he will bestow some attention upon my conclusions, and direct upon them the light of discussion.

Remarks of M. Faye on the Occasion of M. LeVerrier's Letter.

The unexpected result of these profound researches, resumed for the second time with new elements, cannot fail to make a lively impression upon astronomers, and to promote the new examinations which M. LeVerrier himself suggests with an urgency, the motives for which are so fully explained. As one of the hypotheses on which the learned author appears to rest, to account for the motion of the perihelion of Mercury, points to an almost immediate verification, in which observers will first interest themselves, I shall ask the permission of the Academy to submit from this time a sort

of plan of operation. I refer here to the probable existence of a series of small planets within the orbit of Mercury.

A new planet has often been sought for in these dazzling regions, but at a venture, and always in vain. The failure proves nothing, because these researches were merely fanciful. Under the impulse of a weighty probability, the result might be altogether different, provided the operation was conducted upon a plan rationally conceived. And, in the first place, it is evident that the brilliancy of the heavens in the circumsolar region would only have allowed the discovery in this manner of a star of the order of Mercury itself, and not the small planets indicated by M. LeVerrier. We are, then, first led to turn to account the darkness of total eclipses, and particularly that of July next, which is about to afford us an opportunity of trying a first experiment. We know, it is true, that during the greater part of these eclipses we see with the naked eye scarcely anything but the planets and the most brilliant stars. But this fact is explained in a great measure by the duration of the glare. If the observer, instead of following the Sun up to the last moment, remained in darkness a quarter of an hour before the total eclipse, his eye would be much more sensitive at the decisive moment. Let us suppose that an astronomer takes charge of this research at one of the stations of Spain * or of Algeria, where we shall go next year to observe this magnificent phenomenon; let us suppose, moreover, that he is furnished with a good comet-seeker mounted as an equatorial or a theodolite, so as to determine at necessity a direction with a certain degree of accuracy; let us suppose, finally, that he gives up the pleasure of observing the most curious phases, and that he keeps for some time in nearly complete

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* The station of Campvey, already celebrated through the geodetic labors of MM. Biot and Arago, would offer some advantages in this particular.

darkness: he will be entirely ready to catch the least sparkle in the circumsolar region, beyond the aureola; and the few minutes' duration of the total eclipse would be sufficient for him to explore a large part of the region designated by M. LeVerrier.

Total eclipses are sufficiently frequent to prevent a successful observation from remaining a long time isolated; if, then, the result of the next eclipse is not negative, I do not doubt but that we shall very soon be able to obtain some accurate notions concerning the bodies that will be discovered under these particular circumstances.

Be it as it may with regard to this expectation, I have thought it my duty to insist upon a condition of success which has always been neglected, that of avoiding the duration of the dazzle caused by the direct observation of the Sun, or even by the simple effect of daylight.

The learned Director of the Observatory himself points out another method quite as effectual, perhaps. It consists in following with care the small spots with which the Sun is frequently strewed. Planets, of which the orbit would be very little inclined to that of Mercury would have, in fact, like the latter, their transits across the Sun; transits which for a long time might escape the attention of astronomers not forewarned, as did a certain satellite of Saturn in a region as limited and not less explored. But the difficulty even of these researches induces me to return to a suggestion of Sir John Herschel which I have very often called to mind. If, in several observatories suitably selected, pains were taken to photograph the Sun many times a day, with the aid of a large instrument, we should obtain almost a continuous history of the disc of this body, and no one of the phenomena to which M. LeVerrier has just alluded could escape the observer. I have myself shown how the value of an astronomical observation might be given to

these photographs, independently of all apparatus of measure, by taking two impressions upon the same plate, at an interval of two minutes. The beautiful trials of the eclipse of the 15th of March have given beforehand an assurance of success. It will be sufficient to superimpose transparent negatives of this size, taken at intervals of a quarter of an hour, to distinguish immediately the movable projection of an asteroid in the midst of the most complicated groups of small spots. This undertaking would not in any manner debar the usual investigations upon the solar disc, and would retain, even if it should fail for the inter-Mercurial planets, an immense value for the physical history of the Sun itself.

APPLICATION OF THE BINOMIAL THEOREM TO THE EXTRACTION OF THE ROOTS OF WHOLE NUMBERS.

By Theodore Strong, Professor of Mathematics and Natural Philosophy in Rutgers College, New Brunswick, N. J.

If a, b, c, &c. stand for the successive digits of the number, and 10^m represents the local value of a; then the number will be represented by the forms

$$(1)\quad a10^m+b10^{m-1}+c10^{m-2}+\&c.=a10^m\left(1+\frac{b10^{-1}+c10^{-2}+\&c.}{a}\right)=N,$$

noticing that instead of (1) it will sometimes be important, especially when a is greater than 5, to write

$$10^{m+1}-[10^{m+1}-(a\,10^m+b\,10^{m-1}+\&c.)]=N,$$

which is easily changed to the form

$$(2)\qquad 10^{m+1}[1-(a'\,10^{-1}+b'\,10^{-2}+\&c.)]=N.$$

Since

$$1+\frac{b\,10^{-1}+c\,10^{-2}+\&c.}{a}\quad\text{and}\quad 1-\frac{a'\,10^{-1}+b'\,10^{-2}+\&c.}{1}$$

are of the forms

$$1 + \frac{B}{A}, \text{ and } 1 - \frac{C}{D},$$

they may be reduced to products of the forms

$$\left(1 + \frac{B'}{A'}\right)\left(1 + \frac{B''}{A''}\right) \text{ \&c.,} \quad \left(1 - \frac{C'}{D'}\right)\left(1 - \frac{C''}{D''}\right) \text{ \&c.,}$$

such that

$$\frac{B'}{A'}, \frac{B''}{A''}, \text{ \&c.,} \quad \frac{C'}{D'}, \frac{C''}{D'}, \text{ \&c.,}$$

may be of any required degree of smallness.*

Again, since

$$2 = 1 \div \tfrac{1}{2} = 1 \div (1 - \tfrac{1}{2}) = (1 - \tfrac{1}{2})^{-1},$$

we have

$$\begin{array}{ll} 3 = (1 - \frac{1}{2})^{-2}(1 - \frac{1}{4}), & 4 = (1 - \frac{1}{2})^{-2}, \\ 5 = (1 - \frac{1}{2})^{-2}(1 + \frac{1}{4}), & 6 = (1 - \frac{1}{2})^{-3}(1 - \frac{1}{4}), \\ 7 = (1 - \frac{1}{2})^{-3}(1 - \frac{1}{8}), & 8 = (1 - \frac{1}{2})^{-3}, \\ 9 = (1 - \frac{1}{2})^{-3}(1 + \frac{1}{8}), & 10 = (1 - \frac{1}{2})^{-3}(1 + \frac{1}{4}), \text{ \&c.} \end{array}$$

These equations show that the digit a can be reduced to binomial forms, such that the second terms of the binomials shall not numerically exceed the fraction $\frac{1}{2}$; and that for 10^m we may write $(1 - \frac{1}{2})^{-3m}(1 + \frac{1}{4})^m$. From the binomial theorem we readily get the forms

$$(1 + x)^{\frac{r}{n}} = 1 + \frac{r}{n}x + \frac{\frac{r}{n}\left(\frac{r}{n} - 1\right)}{1\,.\,2}x^2 + \frac{\frac{r}{n}\left(\frac{r}{n} - 1\right)\left(\frac{r}{n} - 2\right)}{1\,.\,2\,.\,3}x^3 + \text{ \&c.}$$

$$(3) \qquad = 1 + \frac{r}{n}x + \frac{r(r - n)}{1\,.\,2\,n^2}x^2 + \frac{r(r - n)(r - 2n)}{1\,.\,2\,.\,3\,n^3}x^3 + \text{ \&c.,}$$

in which x and r may be positive or negative.

Examples.

1. "To extract the 100th root of 100."

Because $100 = 10^2$ we have

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* See Dr. Strong's Algebra, page 288. — Ed.

$$\sqrt[100]{100} = 10^{0.02} = (1 - \tfrac{1}{2})^{-0.06} (1 + \tfrac{1}{4})^{0.02} = (1 - 0.5)^{-0.06} (1 + 0.25)^{0.02};$$

consequently, putting — 0.5 for x and — 0.06 for $\frac{r}{n}$, the first form of (3) gives

$$\begin{aligned}(1 - 0.5)^{-0.06} = 1 &+ 0.03 + 0.00795 + 0.0027295 + 0.001044033 + 0.000423877 \\ &+ 0.000178735 + 0.000077364 + 0.000034136 + 0.000015285 \\ &+ 0.000006924 + 0.000003166 + 0.000001458 + 0.000000676 \\ &+ 0.000000315 + 0.000000154 + 0.0000000724 + \text{\&c.} = 1.042465746,\end{aligned}$$

the first seven decimals of which are correct.

Similarly, putting 0.25 for x and 0.02 for $\frac{r}{n}$, the first form of (3) gives

$$\begin{aligned}(1 + 0.25)^{0.02} = 1 &+ 0.005 - 0.0006125 + 0.000101062 - 0.000018823 + 0.000003746 \\ &- 0.000000777 + 0.000000165 - 0.000000036 + \text{\&c.} = 1.004472837,\end{aligned}$$

the first seven decimals being correct.

Hence,

$$\sqrt[100]{100} = 1.04246574 \times 1.004472837 = 1.04712852,$$

the first seven decimal places of which are correct.

2. "To extract the fifth root of 281950621875."

Since the number equals

$$10^{11} \times 2.81950 \text{ \&c.} = 10^{10} \times 4 \times 5 \times 1.4097 \text{ \&c.},$$

we immediately get

$$\begin{aligned}\sqrt[5]{28195 \text{ \&c.}} &= 10^2 (1 - \tfrac{1}{2})^{-\frac{2}{5}} (1 - \tfrac{1}{2})^{-\frac{2}{5}} (1 - \tfrac{1}{2})^{-\frac{1}{5}} (1 - \tfrac{1}{2})^{\frac{1}{5}} (1 + \tfrac{1}{4})^{\frac{1}{5}} (1 + 0.4097 \text{ \&c.})^{\frac{1}{5}} \\ &= 200 (1 + \tfrac{1}{4})^{\frac{1}{5}} (1 - 0.29513)^{\frac{1}{5}} = 200 (1 - 0.11891 \text{ \&c.})^{\frac{1}{5}} \\ &= 200 (1 - 0.11891 \text{ \&c.})^{0.2}.\end{aligned}$$

Putting — 0.11891 &c. for x, and 0.2 for $\frac{r}{n}$ in the first form of (3), we get

$$(1 - 0.11891 \text{ \&c.})^{0.2} = 1 - 0.02378 \text{ \&c.} - 0.00056 \text{ \&c.} - \text{\&c.};$$

consequently, we shall have

$$\sqrt[5]{28195 \&c.} = 200\,(1 - 0.02378 \&c. - 0.00056 \&c. - \&c.).$$

Because the preceding series, as the number of terms is increased, gives 195 as the limiting value of $\sqrt[5]{28195 \&c.}$, we infer that 195 is the exact root; indeed, since the excess of nines in the given number and in the product $195 \times 195 \times 195 \times 195 \times 195$ are equal to zero, 195 must clearly be the exact root.

3. "To extract the seventh root of 2, correct to four decimal places."

Because $1 - \frac{1}{2}$ is of the form $1 - \frac{C}{D}$, it is easily resolved into the product $(1 - \frac{1}{4})\,(1 - \frac{1}{3})$; consequently, we shall have

$$\sqrt[7]{2} = (1 - \tfrac{1}{3})^{-\frac{1}{7}}\,(1 - \tfrac{1}{4})^{-\frac{1}{7}}.$$

Putting $-\frac{1}{3}$ for x, and -1 and 7 for r and n in the second form of (3), we get

$$(1-\tfrac{1}{3})^{-\frac{1}{7}} = 1 + 0.047619 + 0.009070 + 0.002159 + 0.000565 + 0.000156 + 0.000044 + \&c. = 1.059603;$$

and in like manner

$$(1 - \tfrac{1}{4})^{-\frac{1}{7}} = 1 + 0.035714 + 0.005102 + 0.000911 + 0.000160 + 0.000038 + \&c. = 1.041945.$$

Hence, we shall have

$$\sqrt[7]{2} = 1.059603 \times 1.041945 = 1.104048 +,$$

the first four decimals being correct.

Remark. — We have resolved $1 - \frac{1}{2}$ into factors for the purpose of obtaining series of more rapid divergency than the development of $(1 - \frac{1}{2})^{-\frac{1}{7}}$ will give.

Mathematical Monthly Notices.

Logarithmic Tables of Numbers and Trigonometrical Functions. By Baron von Vega. Translated from the Forty-third, or Dr. Bremiker's thoroughly revised and enlarged edition, by W. L. F. Fischer. Stereotyped. One volume octavo. Berlin: Weidmanns. 1859. New York: B. Westermann & Co. Price, $2.25.

Dr Bremiker's edition of Vega's Logarithmic and Trigonometrical Tables of seven decimal figures is unquestionably the best work of the kind that has yet been published. It is a "table portative," of the same order (and about the same size) as that of Callet, — so long popular on account of its superiority over preceding works, — but far surpasses that work in every requisite of a convenient table. 1st. In the clearness of the type. 2d. In the arrangement and grouping of the logarithms on the page, so as to cause the least fatigue to the eye in tracing a line across the page. 3d. In the facility of interpolating afforded by the tables of proportional parts in the trigonometrical portion, which are not given at all in Callet. 4th. In the accuracy with which the seventh place of decimals is determined in consequence of the preservation of the eighth place in the tables of proportional parts. 5th. In the improved arrangement of the table of log. sines, and tangents to every second for the first five degrees of the quadrant. 6th. In the very simple and convenient, as well as extremely accurate, means of obtaining the log. sines, and tangents of small angles (those less than 2° 46′ 40″ or 10,000″), afforded by the numbers marked S and T in the margin of the tables of logarithms; and, lastly, in the accuracy with which the tabular logarithms themselves are given. In this last respect, the tables of Callet have possessed a high reputation, as they were stereotyped as long ago as 1795, and are presumed to have been corrected in the stereotype plates whenever an error has been discovered. But Dr. Bremiker has entered into a critical examination of the seventh decimal place, carefully comparing the logarithms with those of Vega's *Thesaurus Logarithmorum* of ten figures, and recomputing to fifteen places all those logarithms of numbers in the Thesaurus which terminated in 500, in order to decide whether the seventh figure should be increased by a unit; and, for the same purpose, recomputing to fifteen places all the logarithms in the trigonometrical portion which terminated in 496, 497, 498, 499, 500, 501, 502, 503, and 504. A number of errors were thus detected, which have run through all tables down to the present time.

It was a happy idea to publish the tables with an English preface and introduction, the tables themselves being printed from the same stereotype plates as the original German edition, thus making them available to English readers without incurring renewed risks of errors in reprinting, and at the same time keeping the price of the book within the means of everybody.

While the work is admirably suited to the wants of the practical computer, we would also commend it especially to the student who desires to possess at least one good logarithmic table.

The Lady's and Gentleman's Diary, or Poetical and Mathematical Almanac for the Year 1860. London: Printed for the Company of Stationers. Price 1s. 4d.

This is one of the most valuable numbers of "The Diary." All the solutions of the problems proposed in 1859 are excellent, and the new problems are unusually interesting. We are glad to see the solutions of the Prize Question for 1859, by Messrs. Simon Newcomb and G. B. Vose, in the number before us. We append the Prize Question for this year.

Prize Question. — "If an hypocycloid has the radius of its describing circle one third of that of its base, being thus composed of three branches, and a straight line equal to twice the

diameter of the small circle be placed with its extremeties on two of the branches, it will touch the third branch."

A Treatise on Elementary and Higher Algebra. By THEODORE STRONG, LL.D., Professor of Mathematics and Natural Philosophy in Rutgers College, New Brunswick, N. J., Member of the American Philosophical Society, and of the American Academy of Arts and Sciences. New York: Pratt, Oakley, & Co., No. 21 Murray St. 1859.

Our readers will not need to be informed that DR. STRONG has devoted a long life to the successful study of the Mathematics, and that his name will ever be honorably identified with the progress of the science in this country. In the volume before us we find many most valuable results of this study which must be considered as positive additions to the science; and we regret that our space prevents more than a simple reference to some of the more important ones.

The roots of an equation are developed into series, by which it is readily shown that every equation of the nth degree must have n roots. This difficult and important proposition is thus brought within the domain of simple algebra. The solutions of Binomial Equations and the irreducible case of Cubic Equations are effected by pure algebra. A new and general method is given for the determination of the roots of equations. These are among the more important additions; but we would also call attention to the use of Detached Coefficients; to the Appendix to Multiplication and Division; to the Fundamental Principle of Single and Double Position, and of the Differential Calculus with applications; to a new investigation of the Binomial Theorem; to a new method of solving Quadratics, &c.

This is no ordinary work, and to those who like rigorous demonstration and thorough investigation we heartily commend it. We have but one disparaging remark to make. The arrangement of the matter is bad; the equations, prominent as well as less important, are printed and numbered in the text, often causing trouble to find the reference. The ease with which the mathematics are read depends much more upon arrangement than might at first seem; for the logical steps may often, to a great extent, be exhibited to the eye in the arrangement.

The Surveyor's and Engineer's Companion; being a concise Treatise on Mathematical Instruments, containing an improved method of telescopic measurements, illustrated with numerous engravings; and including the most important and useful tables and formulæ constantly used in surveying and engineering. By WILLIAM SCHMOLZ, Mathematical-Instrument Maker, San Francisco.

This is a small pocket field-book of 62 pages. The first part is mainly devoted to the descriptions of mathematical and scientific instruments, with rules for their adjustment and use, a large proportion of instruments being illustrated with very good wood engravings. The second part, prepared by Dr. R. C. MATHEWSON, U. S. Deputy Surveyor, San Francisco, can best be described by giving his full title-page. "A new set of Practical Tables, useful in Surveying and Engineering, containing Easy and Accurate Methods for finding the Variation of the Magnetic Needle at any Hour of the Night, Latitudes and Longitudes of Places from their Differences of Latitude and Departure, the Convergencies of the Meridians, the Divergencies of the Parallels of Latitude and Prime Verticals, Altitudes by the Barometer, Atmospheric Refraction, &c., together with *an Improved Method of Tabling*, which facilitates the Computation of Areas and the Projection of Maps."

The small size of this volume, made so by a judicious selection of matter, strongly recommends it.

Annual Report of the Board of Regents of the Smithsonian Institution, showing the Operations, Expenditures, and Condition of the Institution, for the Year 1858. Washington: James B. Steedman, Printer. 1859.

This Report begins, as usual, with that of the Secretary, Prof. Henry, giving an account of the doings of the Institution during the year, embracing, besides the statement of its scientific operations and publications, an account of its Magnetic Observatory, Laboratory, System of Exchanges, Library, Museum, Gallery of Art, and Lectures. We are glad to learn that a catalogue of the series of Transactions and Proceedings of Learned Bodies now in the library of the Institution, prepared under the direction of the late Prof. W. W. Turner, the learned and accomplished librarian of the Patent Office Library, is soon to be distributed. In the Appendix, "the object of which is to illustrate the operations of the Institution by the reports of lectures and extracts from correspondence, as well as to furnish information of a character suited especially to the meteorological observers and other persons interested in the promotion of knowledge," we find four lectures on astronomy; a memoir of Priestley; instruction for collecting insects; grasshoppers and locusts of America; means of destroying the grasshopper; vegetable colonization of the British Isles; causes which limit vegetable species towards the north, in Europe, and similar regions; forests and trees of North America; list of birds of Nova Scotia; atmospheric electricity; recent progress in physics; meteorology, and correspondence. We wish especially to call attention to the admirable lectures on astronomy, by Prof. Alexis Caswell. They are indeed model "popular lectures," and a more clear and accurate notion of "the figure and magnitude of the earth," "the law of gravitation," "the dimensions of the solar system, or the extent of our knowledge of planetary distances," and "sidereal astronomy," can hardly be desired. They might be made the basis of a valuable text-book on astronomy.

A Treatise on the Higher Plane Curves; intended as a sequel to a Treatise on Conic Sections. By Rev. George Salmon, M. A., Fellow and Tutor, Trinity College, Dublin. Dublin: Hodges and Smith, Grafton Street, Booksellers to the University. 1852.

Our attention has recently been recalled to the above volume by a request from a correspondent that we would recommend a good work on the subject of which it treats.

We cannot better express our high opinion of the work before us than by giving the remarks of James Henthorn Todd, D.D., President of the Royal Irish Academy, upon the occasion of the presentation of the Cunningham Medal to Mr. Salmon, May 24, 1858.

The President said: — "The chief merit of Mr. Salmon's 'Treatise on the Higher Plane Curves' is the clear and full exposition of all the modern improvements in the methods of analytical research which it contains. The author does not profess to have made any new discoveries, or to suggest new methods of investigation, but he has done both; and this new matter is introduced with so little parade, or, I should rather say, generally without any notice at all, that it requires considerable knowledge of the subject to distinguish the discoveries of Aronhold, or Plücker, of Poncelet, or Joachimsthal, from the new and highly interesting propositions introduced whilst giving an account of the investigations of these authors, and connected with their research, but due altogether to Mr. Salmon.

"The method of investigating the properties of conic sections by reference to two tangents and the line joining their points of contact, and the analogous method applied to cubics of the third class; the application of the theory of determinates to the discovery of the properties of curves, particularly to finding the reciprocals of curves of the third and fourth degrees; the investigation of the focal properties of cubical biquadratic curves, and many other new properties of conics, may be mentioned as original, the result of Mr. Salmon's genius and research. But, however valuable and interesting these additions to what previous writers had discovered, the great practical value of the work is, that it arranges in a clear and connected system all the important geometrical discoveries hitherto published, so that, to use the author's own words, 'each new student who wishes to devote himself to original investigation in any branch of

mathematics, may have his energies brought to bear upon the undiscovered parts of the science.' Such a student, with the aid of Mr. Salmon's book, without the labor of searching the scattered papers in scientific journals, or transactions of societies, will at once see what has been already done, and will escape the danger of wasting his abilities by rediscovering what others had discovered long before.

"It would be premature to speak of Mr. Salmon's investigations in the geometry of three dimensions, as they have not as yet been fully given to the public; but we have already had a foretaste of what may be expected from him in this higher region of mathematical research, as he has already read to the Academy a valuable paper on the 'Reciprocal of a Surface of the Second Degree,' and he has contributed to different periodical journals most important investigations relative to the surface of the third degree; all of which, with many additions, we may look forward to, in a collected form, in a third volume of his Geometry, devoted to the properties of surfaces."

Editorial Items.

The following gentlemen have sent us solutions of the Prize Problems in the January number of the Monthly.

M. K. Bosworth, Marietta College, Ohio. Probs. III., IV., and V.

Asher B. Evans, Madison University, N. Y. Probs. I., II., III., IV., and V.

Cadet H. S. Wetmore, U. S. M. Academy. Prob. I.

Gustavus Frankenstein, Springfield, Ohio. Probs. III., IV., and part of V.

T. E. Tower, Amherst College. Probs. III. and IV.

E. S. Stearns, Chester Institute, N. J. Probs. I. and II.

W. H. Spencer, C. J. Baldwin, and H. A. Wood, Madison University Grammar School. Each Prob. I.

T. S. Hubbard, Alfred Centre, N. Y. Probs. I. and II.

W. F. Osborne, Wes. Univ., Middletown, Ct. Probs. III., IV., and V.

D. G. Bingham, Ellicotteville, N. Y. Probs. II. and III.

D. M. Hudson, Paris, Ind. Probs. I., II., and III.

Jas. M. Ingalls, Delton, Sauk Co., Wis. Probs. I., II., and III.

James F. Roberson, Ind. Univ. First part of Prob. V.

J. K. Upton, New London Inst., N. H. Probs. I., II., and III.

We have not given in this number of the Monthly the Method of Integration by Quadratures, as we intended, in order to make room for the very interesting and important communication from Commander Davis on the question of the inter-mercurial planets. We will simply add that, from the chart of the total eclipse of July 18, 1860, just issued from the Office of the Nautical Almanac, we learn that Astoria, Olympia, Fort Okonagon, and Fort Colville, on our northwest coast, Fort York on Hudson's Bay, and the northern point of Labrador, are within the limits of the total phase; and that the eclipse ought to be observed from one or more of these points, having in view, to settle if possible, the question discussed in the communication to which we have referred.

Erratum. — In the February No., last equation of Prize Problem V., in the denominators for $\cos\rho'$, &c., read $\sin\rho'$, &c.

THE

MATHEMATICAL MONTHLY.

Vol. II. . . . MAY, 1860. . . . No. VIII.

PRIZE PROBLEMS FOR STUDENTS.

I. If x is a whole number, prove that $x^3 - x$ is always divisible by 6.

II. Describe a series of circles whose areas shall be one half, one fourth, one eighth, &c. that of a given circle. — Communicated by George Stuart, Haverford College, Pa.

III. If R and r be the radii of the circles circumscribing and inscribing any triangle, and D the distance between their centres, then $D^2 = R^2 - 2\,R\,r$. Required, a geometrical demonstration. — Communicated by George Eastwood, Saxonville, Mass.

IV. If a circle cut a conic section in four points, A, B, C, D, and a second circle cut the same conic section in A, B, E, F, then will $C\,D$ and $E\,F$ be parallel. — Communicated by Asher B. Evans, Madison University, Hamilton, N. Y.

V. In any conic section let u denote the length of the perpendicular dropped from any point in the curve upon the directrix, r the distance of this point from the nearest focus, a and b the semi-axes; prove that

$$\frac{u^2 - r^2}{u^2} = \frac{b^2}{a^2}.$$

— Communicated by David Trowbridge, Perry City, N. Y.

Solutions of these problems must be received by July 1st, 1860.

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. V., Vol. II.

The first Prize is awarded to David Trowbridge, Perry City, Schuyler Co., N. Y.

The second Prize is awarded to George B. Hicks, Cleveland, Ohio.

The third Prize is awarded to Miss Harriet S. Hazeltine, Worcester, Mass.

Prize Solution of Problem III.

By G. B. Hicks and David Trowbridge.

III. Find the n quantities $x_1, x_2, \ldots\ldots x_n$, from the n equations,

$$(1) \qquad x_1 + x_2 + x_3 \ldots\ldots + x_n = A,$$
$$(2) \qquad a_1 x_1 + a_2 x_2 + a_3 x_3 \ldots\ldots + a_n x_n = 0,$$
$$(3) \qquad a_1^2 x_1 + a_2^2 x_2 + a_3^2 x_3 \ldots\ldots + a_n^2 x_n = 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$(n) \qquad a_1^{n-1} x_1 + a_2^{n-1} x_2 + a_3^{n-1} x_3 \ldots\ldots + a_n^{n-1} x_n = 0,$$

and obtain symmetrical expressions for x_1, x_2, &c.

If we multiply each of the equations (1), (2), $\ldots\ldots (n-1)$, by a_1, and subtract (2) from the first product, (3) from the second, and so on, we obtain the $n-1$ equations,

$$(1)' \qquad (a_1 - a_2) x_2 + (a_1 - a_3) x_3 \ldots\ldots + (a_1 - a_n) x_n = A a_1,$$
$$(2)' \qquad a_2 (a_1 - a_2) x_2 + a_3 (a_1 - a_3) x_3 \ldots\ldots + a_n (a_1 - a_n) x_n = 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$(n-1)' \quad a_2^{n-2}(a_1 - a_2)x_2 + a_3^{n-2}(a_1 - a_3)x_3 \ldots\ldots + a_n^{n-2}(a_1 - a_n)x_n = 0,$$

from which x_1 is eliminated.

If now we multiply each of these equations but the last by a_2, and from the first product subtract $(2)'$, from the second $(3)'$, and so on, we shall get $n-2$ equations from which x_2 will be eliminated. Operating on these $n-2$ equations, and then on those which result in a similar manner, and so on, we shall finally obtain

$$(a_1 - a_n)(a_2 - a_n) \ldots\ldots (a_{n-1} - a_n) x_n = A a_1 a_2 \ldots\ldots a_{n-1}.$$

$$\therefore\ x_n = \frac{A\, a_1 a_2 \ldots\ldots a_{n-1}}{(a_1 - a_n)(a_2 - a_n) \ldots\ldots (a_{n-1} - a_n)}.$$

Since a precisely similar process will give any other value of x, it is plain that all the values may be obtained from this one by permutation. The solutions by ASHER B. EVANS and T. E. TOWER are essentially the same as the one here given.

PRIZE SOLUTION OF PROBLEM IV.

By G. B. HICKS and GUSTAVUS FRANKENSTEIN.

IV. Prove that $\sin^n(\theta - \varphi)\sin\varphi$ is a maximum when

$$\sin(\theta - 2\varphi) = \frac{n-1}{n+1}\sin\theta;$$

θ being a given constant, and φ the variable.

If a function is a maximum or a minimum its logarithm will be also; and therefore we may put

$$u = n\log\sin(\theta - \varphi) + \log\sin\varphi.$$

Taking the derivative

$$\frac{du}{d\varphi} = -\frac{n\cos(\theta - \varphi)}{\sin(\theta - \varphi)} + \frac{\cos\varphi}{\sin\varphi} = 0,$$

(1) $$\therefore\ \sin(\theta - \varphi)\cos\varphi = n\cos(\theta - \varphi)\sin\varphi.$$

Now adding $\cos(\theta - \varphi)\sin\varphi$ to both sides of (1), and then subtracting it from both sides, we get

$$\sin\theta = (n+1)\cos(\theta - \varphi)\sin\varphi,$$

and $$\sin(\theta - 2\varphi) = (n-1)\cos(\theta - \varphi)\sin\varphi.$$

$$\therefore\ \frac{\sin\theta}{n+1} = \frac{\sin(\theta - 2\varphi)}{n-1}; \qquad \text{or } \sin(\theta - 2\varphi) = \frac{n-1}{n+1}\sin\theta.$$

Taking the derivative of $\frac{du}{d\varphi}$ gives

$$\frac{d^2u}{d\varphi^2} = -\left(n\,\mathrm{cosec}^2(\theta - \varphi) + \mathrm{cosec}^2\varphi\right),$$

which is essentially negative so long as n is positive, and therefore the given function is a maximum.

Prize Solution of Problem V.

By M. K. Bosworth, Marietta College, Ohio.

V. The notation of Problem V. in the November No. being retained, prove that in the plane triangle

(1) $$\frac{1}{\rho'}+\frac{1}{\rho''}+\frac{1}{\rho'''}=\frac{1}{\rho};$$

and in the spherical triangle,

(2) $$\frac{1}{\tan\rho'}+\frac{1}{\tan\rho''}+\frac{1}{\tan\rho'''}=\frac{1}{\tan\rho}\,\frac{\cos(\rho+\delta)\cos(\rho-\delta)}{\cos^2 r\cos^2\rho};$$

(3) $$\frac{\cos\delta'}{\sin\rho'}+\frac{\cos\delta''}{\sin\rho''}+\frac{\cos\delta'''}{\sin\rho'''}=\frac{\cos\delta}{\sin\rho}.$$

In the plane triangle (Chauvenet's Trig. Eqs. 289 and 293)

$$\frac{1}{\varrho}=\frac{s}{k},\quad \frac{1}{\varrho'}=\frac{s-a}{k},\quad \frac{1}{\varrho''}=\frac{s-b}{k},\quad \frac{1}{\varrho'''}=\frac{s-c}{k},$$

and therefore (1) is true. In the spherical triangle

$$\frac{1}{\tan\varrho}=\frac{\sin s}{k},\ \frac{1}{\tan\varrho'}=\frac{\sin(s-a)}{k},\ \frac{1}{\tan\varrho''}=\frac{\sin(s-b)}{k},\ \frac{1}{\tan\varrho'''}=\frac{\sin(s-c)}{k};$$

$$\therefore\quad \frac{1}{\tan\varrho'}+\frac{1}{\tan\varrho''}+\frac{1}{\tan\varrho'''}=\frac{\sin(s-a)+\sin(s-b)+\sin(s-c)}{k}.$$

But from *Math. Monthly*, Vol. II., p. 229, and $\tan\varrho$, we have

$$\frac{1}{\tan\varrho}\,\frac{\cos(\varrho+\delta)\cos(\varrho-\delta)}{\cos^2 r\cos^2\varrho}=\frac{\sin(s-a)+\sin(s-b)+\sin(s-c)}{k};$$

and therefore (2) is true.

Also, from *Math. Monthly*, Vol. II., p. 150; Chauvenet's Trig., p. 251, and Eqs. 319 and 323, we get

$$\frac{\cos\delta'}{\sin\varrho'}=\frac{-\sin a+\sin b+\sin c}{2k}\cos r,$$

$$\frac{\cos\delta''}{\sin\varrho''}=\frac{\sin a-\sin b+\sin c}{2k}\cos r,$$

$$\frac{\cos\delta'''}{\sin\varrho'''}=\frac{\sin a+\sin b-\sin c}{2k}\cos r.$$

$$\therefore\ \frac{\cos\delta'}{\sin\varrho'}+\frac{\cos\delta''}{\sin\varrho''}+\frac{\cos\delta'''}{\sin\varrho'''}=\frac{\sin a+\sin b+\sin c}{2k}\cos r=\frac{\cos\delta}{\sin\varrho}.$$

Simon Newcomb.
W. P. G. Bartlett.
Truman Henry Safford.

NOTES AND QUERIES.

1. *Note on Co-Factors.*—If one factor of a product is known, the other may often be found by inspection. For example, 179 being one factor of 17363, the other factor must be 97, because 7 is the only digit that, when multiplied by 9, gives a product terminating in 3. If the product is an odd number, and not divisible by 5, the right-hand figure of the unknown factor may always be determined, and the left-hand figure may be ascertained as in ordinary division.

If the factor sought contains three figures, the middle figure may generally be found by *casting out nines.*

Let $9n + a$, and $9n_1 + x + y$, be the two factors of any given number $9n_2 + b$. Multiplying the two factors together, we find that $a(x + y) = 9n_3 + b$. The values of a and b being known, the value of $x + y$ may be ascertained by substituting for n_3 the successive values 0, 1, 2, &c., until we obtain a value for $9n_3 + b$ that is divisible by a, and adding $9n_4$ to the quotient, (n_4 being equal to either 0, 1, or 2,) x may be considered as equivalent to the sum of the right-hand and left-hand digits of the unknown factor, and $y = \frac{9n_3 + b}{a} - x$, will then be the middle digit.

Let it now be required to determine by inspection the value of z, in the equation $9769z + 471 = 8333428$. Transposing, $9769z = 8332957$. Rejecting the 9's from 9769 we obtain $a = 4$; rejecting the 9's from 8332957 gives $b = 1$. The least value of n_3, that makes $9n_3 + b$ divisible by 4, is 3; $\therefore x + y = 9n_4 + 7$. The left-hand digit is evidently 8, and the right-hand digit 3;

$$\therefore x = 11; \qquad y = 5; \qquad z = 853.$$

If $a = 0$, y may have ten possible values (from 0 to 9 inclusive). If $a = 3$, or 6, y may have three possible values. For any other value of a there can be only one possible value of y.

Table I. will give the right-hand digit, except when $a = 0$, 3, or 6. The value of a is to be sought in either margin, b in the body of the table, on a line with a, and $x + y - 9n_4$ in the other margin on a line with b.

TABLE I.

	1	2	4	5	7	8
1	1	2	4	5	7	8
2	2	4	8	1	5	7
4	4	8	7	2	1	5
5	5	1	2	7	8	4
7	7	5	1	8	4	2
8	8	7	5	4	2	1

TABLE II.

1	2	4	5	7	8
1, 1	1, 2	1, 4	1, 5	1, 7	1, 8
2, 5	4, 5	2, 2	2, 7	2, 8	2, 4
4, 7	7, 8	5, 8	4, 8	4, 4	5, 7
8, 8		7, 7		5, 5	

Table II. gives all the possible forms of the two co-factors ($9n + a$, $9n_1 + a_1$) of any composite number $9n_2 + b$, except when $a = 0$, 3, or 6. The value of b is to be sought in the upper line, and the possible values of a, a_1, will be found in the column below.

Example. — 12697 ($= 9n_2 + 7$) must either be a prime number, or if it have factors, they must be of the form $(9n + 1)(9n_1 + 7)$, $(9n + 2)(9n_1 + 8)$, $(9n + 4)(9n_1 + 4)$, or $(9n + 5)(9n_1 + 5)$. — PLINY EARLE CHASE, Philadelphia.

2. *Proposition in the Theory of Numbers.* — Let any number be separated into periods of three figures each, counting from the right; then if the difference of the sums of the odd (1, 3, 5) and of the even (2, 4, 6) periods be zero, or be divisible by 7, 11, or 13, the number itself will be divisible by 7, 11, or 13.

If we divide the successive powers of 10 by 7, attending to the remainders only, which may be considered either positive or negative, we shall have

$$10 = 7n - 4, \qquad 10^3 = 7n'' - 1, \qquad 10^5 = 7n^{\text{IV}} + 5,$$
$$10^2 = 7n' - 5, \qquad 10^4 = 7n''' + 4, \qquad 10^6 = 7n^{\text{V}} + 1.$$

It is evident that if the division be further continued the remain-

ders will follow each other in the order of the series, -4, -5, -1, $+4$, $+5$, $+1$; and generally, the powers of 10 whose exponents are multiples of 3, and odd, will be of the form $7n-1$, and those whose exponents are multiples of 3, and even, of the form $7n'+1$. Now let there be any number $a\,b\,c\,d\,e\,f\,g\,h\,i\,k$, and let it be separated into periods of three figures. For the sake of convenience, let each period be denoted by a single letter; then

$$a,\, bcd,\, efg,\, hik = lmpq.$$

The number $lmpq$ may be put under the form

$$\begin{aligned} lmpq &= 10^9 l + 10^6 m + 10^3 p + q, \\ &= (7n-1)\,l + (7n'+1)\,m + (7n''-1)\,p + q, \\ &= 7\,(n+n'+n'') - l + m - p + q. \end{aligned}$$

Now the part $7\,(n+n'+n'')$ is divisible by 7, consequently, if the part $-l+m-p+q$ is divisible by 7 the number $lmpq$ is also. But this latter part is equal to the difference of the sums of the odd and even periods into which the given number was separated.

By the same method the proposition may be proved true for the numbers 11 and 13. — H. Willey, Esq., New Bedford, Mass.

3. *Note on the Ox Question.* — Would not the following come more properly under the definition of an *analytical* solution?

"If a oxen in m weeks eat b acres of grass, and c oxen in n weeks eat d acres, how many oxen will eat e acres in p weeks, the grass being supposed to grow uniformly?"

Let $x =$ the required number, α the grass at first on an acre, and β the growth of one acre in one week; then

$b\,(\alpha + m\beta) =$ grass on	b	acres at	end of	m	weeks,
$d\,(\alpha + n\beta) =$ "	d	"	"	n	"
$e\,(\alpha + p\beta) =$ "	e	"	"	p	"

The quantity of grass consumed will be proportional to *both the number of oxen and the time together*; hence

$$(1) \qquad b(\alpha + m\beta) : d(\alpha + n\beta) :: ma : nc,$$

$$(2) \qquad b(\alpha + m\beta) : e(\alpha + p\beta) :: ma : px.$$

$$(1) \text{ gives } \beta = \frac{(ma\alpha - ncb)\alpha}{mn(bc - ad)}, \qquad (2) \text{ gives } x = \frac{mae(\alpha + p\beta)}{pb(\alpha + m\beta}.$$

$$\therefore x = \left(\frac{m-p}{m-n}\right)\frac{nce}{pd} - \left(\frac{n-p}{m-n}\right)\frac{mae}{pb}.$$

Any one of the quantities may be assumed as unknown, and its value obtained from this last equation. — SAMUEL SCHOOLER, Principal of Edge Hill School, Guiney's P. O., Va.

4. *Right-Angled Triangles with commensurable Sides.* — Teachers frequently have occasion to represent the sides of right-angled triangles by exact numbers. Three, four, and five, or equimultiples of those numbers, are often employed, so often in fact that pupils sometimes suppose no others can be used. The following facts may be of service in giving variety to problems, or in illustrating algebraic formulas relating to the right-angled triangle.

There are sixteen dissimilar right-angled triangles with commensurable sides, including only those in which the hypothenuse contains the greatest common measure less than one hundred times. The relations of their sides may be represented by

$$3^2 + 4^2 = 5^2,\quad 5^2 + 12^2 = 13^2,\quad 8^2 + 15^2 = 17^2,\quad 7^2 + 24^2 = 25^2,$$
$$20^2 + 21^2 = 29^2,\quad 12^2 + 35^2 = 37^2,\quad 9^2 + 40^2 = 41^2,\quad 28^2 + 45^2 = 53^2,$$
$$11^2 + 60^2 = 61^2,\quad 16^2 + 63^2 = 65^2,\quad 33^2 + 56^2 = 65^2,\quad 48^2 + 55^2 = 73^2,$$
$$13^2 + 84^2 = 85^2,\quad 36^2 + 77^2 = 85^2,\quad 39^2 + 80^2 = 89^2,\quad 65^2 + 72^2 = 97^2.$$

The greatest common measure, here represented by unity, may be of any length, and hence there may be an infinite number of similar triangles represented by each of the above equations. If unity is supposed to represent an absolute length, as an inch, a foot, or a

yard, then the similar triangles will be represented by like parts and equimultiples of the numbers composing each equation, such as $(\frac{3}{2})^2 + 2^2 = (\frac{5}{2})^2$, and $6^2 + 8^2 = 10^2$. Rejecting all fractions, and all numbers above 100, the first of the above equations gives 20 similar triangles, the second 7, the third 5, the fourth 4, the fifth 3, and the sixth and seventh 2 each, making in all, 52 right-angled triangles, whose sides may be represented by integers not exceeding 100, when unity represents some fixed linear unit. Those represented by the fifth equation approach nearest to the isosceles right-angled triangle, and that represented by the thirteenth equation has the greatest difference between the sides including the right angle.

When the sides of a right-angled triangle are commensurable, the perpendicular let fall from the vertex of the right angle to the hypothenuse is commensurable with the sides, because the two partial triangles thus formed are similar to the original one. In other words, when the sides of a right-angled triangle can be exactly expressed by numbers, the perpendicular can also be exactly expressed, usually by the aid of fractions, as the *greatest* common measure of the sides does not measure the perpendicular. In such a triangle, the diameter of the inscribed circle is also commensurable with the sides, as it is equal to the sum of the other two sides minus the hypothenuse. The perpendicular is equal to the product of the other two sides divided by the hypothenuse.—Prof. D. W. Hoyt, Fairfax, Vt.

5. The method of developing the expression $A = \frac{15\,(E - \sin E)}{9\,E + \sin E}$, employed by Prof. Cotter, in the Mathematical Monthly for March, p. 185, is not the most natural, and does not exhibit the law of progression of the coefficients of the resulting series otherwise than by inspection of the result when obtained. By pursuing the following more direct process, the law of progression is evident at each step.

We have to develop A in terms of T, having $T = \tan^2 \frac{1}{2} E$. We take

$$E = 2 \tan^{-1} T^{\frac{1}{2}} = 2\, T^{\frac{1}{2}} (1 - \tfrac{1}{3} T + \tfrac{1}{5} T^2 - \tfrac{1}{7} T^3 + \tfrac{1}{9} T^4 - \&c.)$$

$$\sin E = \frac{2\, T^{\frac{1}{2}}}{1 + T} \quad = 2\, T^{\frac{1}{2}} (1 + T)^{-1}$$

$$= 2\, T^{\frac{1}{2}} (1 - T + T^2 - T^3 + T^4 - \&c.)\,;$$

whence

$$15\,(E - \sin E) = 30\, T^{\frac{1}{2}} (\tfrac{2}{3} T - \tfrac{4}{5} T^2 + \tfrac{6}{7} T^3 - \tfrac{8}{9} T^4 + \&c.)$$

$$= 20\, T^{\frac{1}{2}} (T - \tfrac{6}{5} T^2 + \tfrac{9}{7} T^3 - \tfrac{12}{9} T^4 + \&c.)$$

$$9\, E + \sin E = 2\, T^{\frac{1}{2}} (10 - \tfrac{12}{3} T + \tfrac{14}{5} T^2 - \tfrac{16}{7} T^3 + \&c.)$$

$$= 20\, T^{\frac{1}{2}} (1 - \tfrac{6}{15} T + \tfrac{7}{25} T^2 - \tfrac{8}{35} T^3 + \&c.)$$

the substitution of which gives the required development,

$$A = \frac{T - \frac{6}{5} T^2 + \frac{9}{7} T^3 - \frac{12}{9} T^4 + \&c.}{1 - \frac{6}{15} T + \frac{7}{25} T^2 - \frac{8}{35} T^3 + \&c.}.$$

— Prof. W. Chauvenet, Washington University, St. Louis, Mo.

6. *Note on the Theory of Perspective.* — The theory of perspective may often be applied to the discovery and demonstration of geometrical truths. I propose to give a few examples of this application. Through the points A, B, C, in the same straight line, let

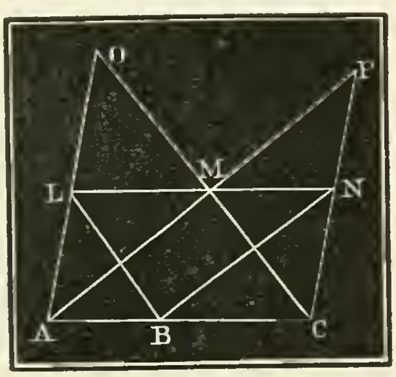


Fig. 1.

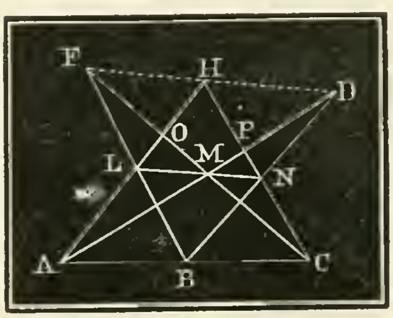


Fig. 2.

three sets of parallels be drawn, as in Fig. 1. The points L, M, N are in the same straight line, for

$$A\,L : L\,O = A\,B : B\,C = P\,N : N\,C.$$

Therefore the line $L\,N$ passes through the intersection of $O\,C$ and

AP. Fig. 2 is the projection of Fig. 1, FD being the horizon, if the parallel lines are horizontal. The points L, M, N are still in the same straight line; whence the following theorem of Apollonius.

"If any quadrilateral be divided into two other quadrilaterals by a straight line, the intersections of the diagonals of the three quadrilaterals are on the same straight line."

As a second example, let us take a circle. Draw any two diameters, and at their extremities draw four tangents. If now we connect the intersections of the diameters and tangents by lines, and also draw tangents at the points where these lines cut the circle, a symmetrical figure will be formed. If we take the perspective of this figure, we may assume any ellipse for the projection of the circle, and any point within it for the projection of the centre. Any line drawn through this point may be taken as the projection of one of the diameters; the projection of the other is then determined by the angle which the original diameters make with each other. The original tangents are evidently tangents to the ellipse in the projection of the figure. We have then this general theorem.

I. Any two lines which are parallel in the original figure meet on the horizon in the perspective representation.

II. If any line is bisected in the original figure, this line prolonged to the horizon is divided harmonically.

Problem V. in the Math. Monthly, Vol. I. No. I., is included in the first part of this theorem. — G. B. Vose, Assistant U. S. Coast Survey, Washington, D. C.

7. *Note on the "Rule of False."* — This rule admits of a very simple explanation, which has probably occurred to most persons who are in the habit of using the rule.

The explanation suggests the origin of the rule, and cannot be new; still I do not remember having seen it.

In any equation of one unknown quantity, such as $f(x) = 0$, let

us for a moment suppose x to be the variable abscissa of a curve, while y, the variable value of $f(x)$, is the ordinate. The equation becomes

$$f(x) = y.$$

We wish to find the point where the curve intersects the axis of x; that is, a value of x which shall make $y = 0$.

Find by trial two values of x, which we may designate by x_1, and x_2, which shall give two values of y, y_1 and y_2, nearly equal to zero. *The rule of false, or rule of double position, is based on the supposition that the curve near the axis of* x *is a right line.*

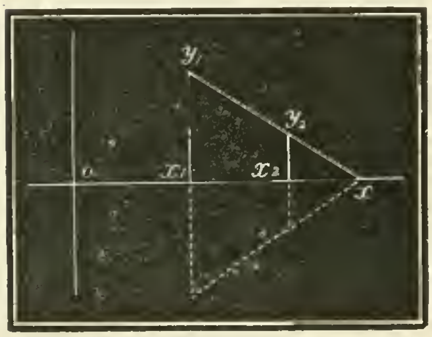


Fig. 1.

$$o\,x_1 = x_1$$
$$o\,x_2 = x_2$$
$$o\,x = x$$
$$x_1\,y_1 = y_1$$
$$x_2\,y_2 = y_2$$

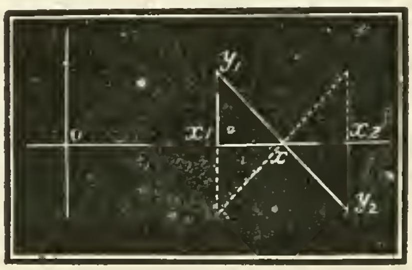


Fig 2.

If y_1 and y_2 are both positive or both negative, Fig. 1 gives

$$y_1 : y_2 :: (x - x_1) : (x - x_2); \qquad \text{or } x = \frac{y_1 x_2 - y_2 x_1}{y_1 - y_2}.$$

If y_1 be positive and y_2 negative, or y_2 positive and y_1 negative, Fig. 2 gives

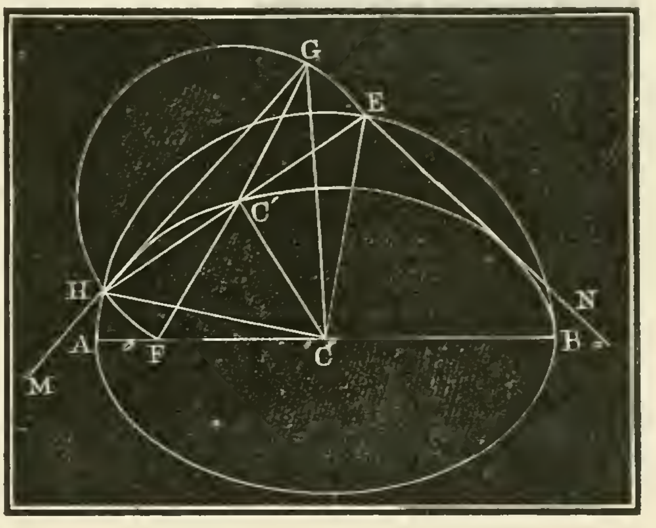
$$y_1 : y_2 :: (x - x_1) : (x_2 - x)\cdot \qquad \text{or } x = \frac{y_1 x_2 + y_2 x_1}{y_1 + y_2}.$$

These two formulæ, enunciated in words, make the "rule of false." It will be seen that the errors, so called, are true ordinates of the curve, and that the final result will be more or less accurate, according to the nature of the curve and the proximity of y_1 and y_2 to zero.

The rule as announced in arithmetics does not apply to the case in which the two "positions," x_1 and x_2, are unlike.

If we put the known term in the second member $f'(x) = p$, the solution will give the point or points where the curve intersects a line parallel to the axis of x at the distance p above it. — Capt. D. P. Woodbury, U. S. A.

9. *Solution of Prize Problem III. in Vol. I. No. V.* — *Problem.* "If two sides of a movable right angle are always tangent to a given ellipse, its summit will describe a circle concentric with the ellipse, the radius of which is equal to the chord joining the extremities of the major and minor axes."



Solution. Let F be the foci of any ellipse, and let GM and GN be the sides of a right angle tangent to it. Join GF, and on it as a diameter describe a segment of a circle, cutting the circle described on AB, the major axis, in E and H. It is evident that GM will pass through H, and GN through E. (See Coffin's Conic Sections, Prop. XV.) And since MGN is a right angle, HE is a diameter to the circle on GF, and hence it passes through C', its centre. From C, the centre of the ellipse, draw CH, CC', CG and CE. Represent the semi-axes as usual by A and B. Then (Legendre, Prop. XIV. B. IV.)

$$2\,HC'^2 + 2\,CC'^2 = CH^2 + CE^2 = 2\,A^2.$$

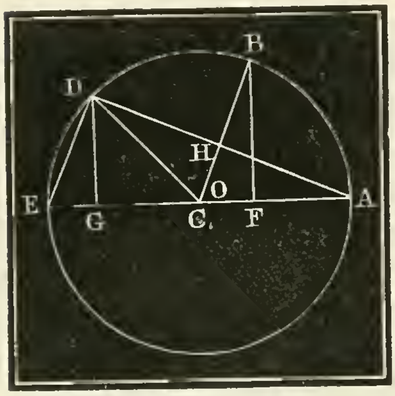
Also, $2\,FC'^2 + 2\,CC'^2 = FC^2 + CG^2 = A^2 - B^2 + CG^2$

Whence, since $HC' = FC'$, $2\,A^2 = A^2 - B^2 + CG^2$.

Therefore $CG = \sqrt{(A^2 + B^2)}$, which was to be proved.

— M. C. Stevens, Prof. of Mathematics, Haverford College, Pa.

10. *Graphic Demonstration of the Formula* $2 \sin \varphi \cos \varphi = \sin 2 \varphi$. — Let $B\,C\,A = \varphi =$ any angle; $A\,C\,D = 2\,\varphi$; $R =$ radius of the circle $A\,B\,E$.



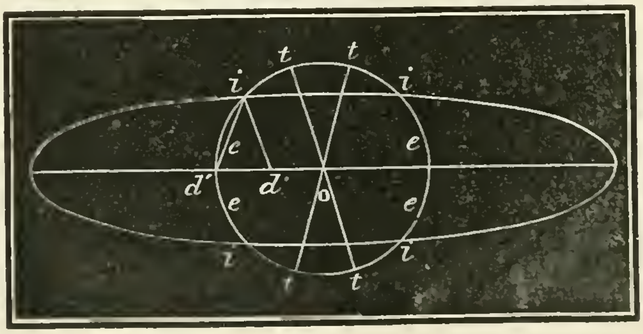
Drop a perpendicular from B to $C\,A$, $B\,C$ is perpendicular to $H\,A$; therefore the right-angled triangles $C\,B\,F$ and $C\,A\,H$ are similar. They are equal, for the hypothenuses $C\,B$ and $C\,A$ are radii. They are also equal to $D\,H\,C$. It follows from this that the triangle $D\,C\,A = 2\;C\,B\,F = B\,F \times F\,C$.

From D drop a perpendicular to $E\,C$; the area of the triangle $E\,C\,D = \frac{1}{2}\,D\,G \times E\,C = \frac{1}{2}\,D\,G \times R$. The area of the triangle $E\,A\,D = \frac{1}{2}\,D\,G \times E\,A = \frac{1}{2}\,D\,G \times 2\,R = D\,G \times R$. But $D\,E\,A - D\,E\,C = B\,F \times F\,C = D\,C\,A = D\,G \times R - \frac{1}{2}\,D\,G \times R = \frac{1}{2}\,D\,G \times R$, therefore $D\,E\,C = D\,C\,A$. But $B\,F = R \sin \varphi$; $C\,F = R \cos \varphi$; $D\,G = R \sin 2\,\varphi$. We have just seen that twice the triangle $B\,C\,F =$ triangle $D\,E\,C$; i. e. $B\,F \times F\,C = \frac{1}{2}\,D\,G \times R$, therefore $R \sin \varphi \times R \cos \varphi = R \times \frac{1}{2}\,R \sin 2\,\varphi$; hence,

$$2 \sin \varphi \cos \varphi = \sin 2\,\varphi.$$

— E. Harrison, St. Louis, Mo.

11. *Note on the Trisection of an Arc.* — An ellipse and concentric circle are so drawn that $B < R$, and $A = B + 2\,R$. Required a geometrical trisection of each of the arcs intercepted on the circumference of the circle by the ellipse.



Solution. — From either of the four points of intersection, i, with a radius $= B$, cut A at d, d'. Through the centre, o, draw $o\,t$ parallel to $d\,i$, or $(d'\,i)$. Then

the exterior arc will be trisected at t. From t, with radius R, cut the circumference of the circle (on the same side of the ellipse's minor axis) at e. Then the interior arc will be trisected at e. The arcs may also be trisected by means of a radius $= A$.

Proof. — Extend id to t. Angle itt and tot are equal, and arc $itt =$ twice tt. $\therefore itti$ is trisected at t and $180° - itti$ is trisected by ie, which $= 60° - it$.

NOTE. — SALMON'S "Treatise on Conic Sections" gives the problem of the trisection of circular arcs, the point of trisection being determined as the intersection of the given arc with a given hyperbola. I have never met with a solution of the problem by means of the ellipse. — PLINY EARLE CHASE, Philadelphia, Pa.

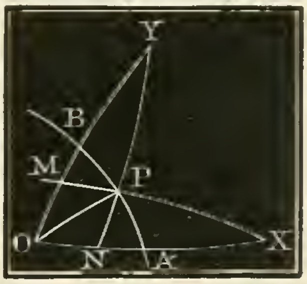
ON SPHERICAL ANALYSIS.

By GEORGE EASTWOOD, Saxonville, Mass.

[Continued from Page 190.]

PROPOSITION IV.

To find the equation of a great circle of the sphere in terms of the co-ordinates of its pole. — Sometimes it is more convenient to use the equation of a great circle expressed in terms of the co-ordinates of its pole, than the one expressed in terms of the intercepts of the axes. Suppose, therefore, that AB, in the annexed diagram, is the proposed great circle; that x_1, $\bar{y}_1$, are the geographical co-ordinates of its pole or *centre*, and that x, $\bar{y}$, are the geographical co-ordinates of any point P in its circumference.



By inspection of the figure, $\frac{1}{2}\pi - \bar{y}_1$, $\frac{1}{2}\pi - \bar{y}$, and $\frac{1}{2}\pi$, are the sides of a spherical triangle whose vertex is at Y, and whose vertical angle is $x_1 - x$. Hence, by spherics, we have

$$\cos\tfrac{1}{2}\pi = \sin\bar{y}\sin\bar{y}_1 + \cos\bar{y}\cos\bar{y}_1\cos(x_1 - x),$$

and by division and expansion, we have

$$\tan\bar{y}\tan\bar{y}_1 = -(\cos x\cos x_1 + \sin x\sin x_1).$$

But, when the axes are rectangular, we have $\tan \bar{y} = y \cos x$, $\tan \bar{y}_1 = y_1 \cos x_1$; the substitution of which in $\tan \bar{y} \tan \bar{y}_1$, gives, after dividing out $\cos x \cos x_1$, $y\, y_1 = -1 - x\, x_1$,

or
$$-\frac{y}{\frac{1}{y_1}} - \frac{x}{\frac{1}{x_1}} = 1.$$

Comparing this equation with $\frac{y}{\beta} + \frac{x}{\alpha} = 1$, we see that $\beta = -\frac{1}{y_1}$, and $\alpha = -\frac{1}{x_1}$.

Hence the equation of a great circle expressed in tangent-functions of the co-ordinates of its pole is

$$y\, y_1 + x\, x_1 + 1 = 0. \tag{22}$$

NOTES ON PROBABILITIES.

By Simon Newcomb, Nautical Almanac Office, Cambridge, Mass.

[Continued from Page 140.]

22. The *a priori* probability of an event is its probability deduced solely from circumstances which preceded or might have preceded it. In the theory of probabilities these circumstances are regarded as causes.

The *a posteriori* probability of an event or hypothesis is its probability deduced from events which have followed it, which events are regarded as its possible effects.

Theorem.—The *a priori* probability of an event is equal to the sum of the products obtained by multiplying the probability of each distinct cause by the probability that the event would occur on the hypothesis that the cause operates.

Let m be the whole number of separate cases. Let n of these cases be favorable to the first cause, n' to the second, &c. Their probabilities will then be $\frac{n}{m}, \frac{n'}{m}, \frac{n''}{m}$, &c. Also, let h of the n cases,

h' of the n' cases, &c., be favorable to the production of the event. Its probability, on the several hypotheses of the occurrence of each separate cause, will then be $\frac{h}{n}$, $\frac{h'}{n'}$, $\frac{h''}{n''}$, &c. Multiplying the corresponding fractions in the two series, and taking the sum of the products we obtain $\frac{h + h' + h'' + \text{\&c.}}{m}$. But this is the probability of the event, because $h + h' + h'' +$ &c. is the whole number of cases favorable to its production.

23. *Problem.* — Let E, E', E'', &c., represent a *portion* of the events which may occur on a certain trial, and let their respective probabilities be p, p', p'', &c. Suppose now that after I have estimated these probabilities, I am informed by a person who knows the result of the trial that one of the events E, E', E'', &c., has actually occurred. What ought to be their respective probabilities in my mind *after I have received this information?*

Let m be the whole number of possible cases; let n of these cases be favorable to E, n' to E', n'' to E'', &c. Then we have

$$p = \frac{n}{m}; \qquad p' = \frac{n'}{m'}, \text{ \&c.}$$

But when it is ascertained that one of the events E has actually occurred, the number of possible cases is *reduced* from m to $n + n' + n'' +$ &c., those favorable to the production of some one of the events E. The probabilities of these events will therefore become

$$\frac{n}{n + n' + \text{\&c.}}, \qquad \frac{n'}{n + n' + \text{\&c.}};$$

that is, each of the former probabilities will be multiplied by the factor $\frac{m}{n + n' + n'' + \text{\&c.}}$; or, *they will each be increased in the same ratio* to such a degree as to make their sum equal to unity.

Example. — On a table are fifty small boxes, of which three each contain a gold coin, seven a silver coin, thirteen a copper coin, and

twenty-seven nothing. The probability that one selected at random contains a gold coin is $\frac{3}{50}$, silver $\frac{7}{50}$, &c. But if by shaking the box it is found that it contains a coin, it will be known that one of the 23 boxes containing coins must have been selected. The probability that the box contains gold will then be increased to $\frac{3}{23}$, that it contains silver to $\frac{7}{23}$, that it contains copper to $\frac{13}{23}$.

24. *Problem.*— To find the probability of an event or hypothesis from both *a priori* and *a posteriori* circumstances.

Suppose that a possible event E, if it occur at all, must be preceded by *one* and *one only* of the causes or circumstances C, C', C'', &c. Let p, p', p'', &c., be the probabilities of C, C', C'', &c., before it is known whether the event E has occurred. Let q be the probability of E on the hypothesis that C occurs, q' its probability on the hypothesis that C' occurs, &c. The *a priori* probabilities of the compound events CE, $C'E'$, &c., will then be pq, $p'q'$, &c. If, now, it is ascertained that the event E has actually occurred, it becomes certain that *one* of these compound events has occurred; and, by § 23, their probabilities will be increased to

$$(1) \qquad \frac{pq}{pq + p'q' + \text{\&c.}}, \qquad \frac{p'q'}{pq + p'q' + \text{\&c.}}, \text{ \&c.}$$

But, the event E being now certain, the probabilities of the causes C, C', &c., are the same as the probabilities of the compound events CE, $C'E'$, &c.; so that the above fractions represent the probabilities of the causes or hypotheses, deduced from all the circumstances in our knowledge, which bear on the question.

Example I.— A, handling a die, entertains a suspicion of $\frac{1}{3}$ that the die is so loaded that a six is as likely to be thrown as not. He throws it twice, and each time throws a six. What is the probability that his suspicion is correct? The solution is exhibited as follows:—

C	p	q	pq
die not loaded,	$\frac{2}{3}$	$\frac{1}{6^2} = \frac{1}{36}$	$\frac{1}{54}$,
die loaded,	$\frac{1}{3}$	$\frac{1}{2^2} = \frac{1}{4}$	$\frac{1}{12}$.

The probability that the die is loaded is therefore

$$\frac{\frac{1}{12}}{\frac{1}{54}+\frac{1}{12}}=\frac{9}{11}.$$

Had he thrown six a third time this probability would have increased to $\frac{27}{29}$.

Example II. — An urn contains 99 white balls, and 1 black one. An individual who falsifies once in ten times draws a ball at random, and asserts that it is black. What is the probability that he tells the truth?

The *a priori* probability that the black ball is drawn, and that he tells the truth respecting it, is $\frac{1}{100}\times\frac{9}{10}$. The *a priori* probability that a white ball is drawn, and that he falsifies, is $\frac{99}{100}\times\frac{1}{10}$. The probability that he tells the truth is therefore

$$\frac{\frac{9}{1000}}{\frac{9}{1000}+\frac{99}{1000}}=\frac{1}{12}.$$

NOTE.* — It is not necessary, as has been supposed by Mr. JOHN STUART MILL, that C and E should be connected by the *philosophical relation* of cause and effect. No such relationship has been supposed in our reasoning. The fraction $p\,q$ does not, as Mr. MILL supposes, express the probability of E on the hypothesis that C is true, but the probability of the compound event $C\,E$. The groundlessness of the former opinion may be shown by solving the following problem. Find the *a priori* probability that it will rain to-morrow, considered as an effect of the planet Jupiter being inhabited or uninhabited.

Let p be the probability that Jupiter is inhabited, r the probability that it will rain to-morrow if Jupiter is inhabited, r also the probability of the same supposing Jupiter to be uninhabited.

By § 22 the probability required will be $p\,r+(1-p)\,r=r$.

This is the common-sense result, so that although we learn nothing new by bringing in irrelevant circumstances as causes, yet neither are we led into error by so doing.

Suppose now that to-morrow arrives, and it does rain; substituting in (1) we have for the probability that Jupiter is inhabited

$$\frac{p\,r}{p\,r+(1-p)\,r}=p.$$

If it does not rain, we have, by a corresponding supposition,

$$\frac{p\,(1-r)}{p\,(1-r)+(1-p)\,(1-r)}=p.$$

* Since this note was in type, we have found that MR. MILL has corrected this error.

THE METHOD OF INTEGRATION, &c.*

The tables for integration at the end of the preceding example consist, besides the dates which limit the intervals of a column of arguments depending on the time from the beginning of the computation, or the number of intervals, a column containing the values of the differential coefficients for the several intervals, a column containing the sum of the series of values of the differential coefficients, and, in the case of the double integral, a column containing a second summed series, or a summing up of the numbers of the first summed series.

The following scheme represents the general form of the table for the double integral: —

Argument.	Principal Function. f.	I. Summed Series. $'f$.	II. Summed Series. $''f$.
$a - 1\,\omega$	$f(a-1)$		$''f(a-1)$
		$'f(a-\frac{1}{2})$	
a	$f\,a$		$''f\,a$
		$'f(a+\frac{1}{2})$	
$a + 1\,\omega$	$f(a+1)$		$''f(a+1)$
		$'f(a+\frac{3}{2})$	
$a + 2\,\omega$	$f(a+2)$		$''f(a+2)$
		$'f(a+\frac{5}{2})$	
$a + 3\,\omega$	$f(a+3)$		$''f(a+3)$
		$'f(a+\frac{7}{2})$	
$a + 4\,\omega$	$f(a+4)$		$''f(a+4)$

ω being the interval, and a the time of beginning, or the date from which the intervals are counted forwards and backwards.

The terms of the two series are derived as follows: —

For the first series,

$$
\begin{aligned}
&'f(a-\tfrac{1}{2}) \\
&'f(a+\tfrac{1}{2}) = f\,a + {'f}(a-\tfrac{1}{2}) \\
&'f(a+\tfrac{3}{2}) = f(a+1) + {'f}(a+\tfrac{1}{2}) \\
&'f(a+\tfrac{5}{2}) = f(a+2) + {'f}(a+\tfrac{3}{2}).
\end{aligned}
$$

* The reader of Professor Encke's treatise upon Mechanical Quadratures will perceive that the following explanation is not a connected translation of any particular part of that treatise, but a summary of the whole, generally made, for the sake of abbreviation, in my own language. — C. H. D.

For the second series,

$$\begin{aligned}
&{}''f(a-1)\\
&{}''f\,a = {}'f(a-\tfrac{1}{2}) + {}''f(a-1)\\
&{}''f(a+1) = {}'f(a+\tfrac{1}{2}) + {}''f\,a\\
&{}''f(a+2) = {}'f(a+\tfrac{3}{2}) + {}''f(a+1)\\
&{}''f(a+3) = {}'f(a+\tfrac{5}{2}) + {}''f(a+2)\\
&{}''f(a+4) = {}'f(a+\tfrac{7}{2}) + {}''f(a+3), \text{\&c.}
\end{aligned}$$

The comprehensive form of the single integral from the middle of the interval preceding the beginning $n = -\frac{1}{2}$ to the middle of any other interval is,

$$\begin{aligned}
\int_{-\frac{1}{2}}^{i+\frac{1}{2}} f(a+n\omega)\,dn = {} & \quad {}'f\,(a+i+\tfrac{1}{2}) - {}'f\,(a-\tfrac{1}{2})\\
& + \tfrac{1}{24}f'\,(a+i+\tfrac{1}{2}) - \tfrac{1}{24}f'\,(a-\tfrac{1}{2})\\
& - \tfrac{17}{5760}f'''\,(a+i+\tfrac{1}{2}) + \tfrac{17}{5760}f'''\,(a-\tfrac{1}{2})\\
& + \tfrac{367}{967680}f^{\mathrm{v}}\,(a+i+\tfrac{1}{2}) - \tfrac{367}{967680}f^{\mathrm{v}}\,(a-\tfrac{1}{2})
\end{aligned}$$

It is evidently a matter of indifference what value is assigned to the term ${}'f(a-\frac{1}{2})$, in the series of the summed functions ${}'f$, since it is first added to every number in forming the series, and afterwards subtracted by the integration, as is seen in the above equation. It may either be taken $= 0$, or equal to the value of the integral up to the limit $n = -\frac{1}{2}$, if the new computation should join on to an earlier. If it be taken $= 0$, and the other terms depending on the limit ${}'f(a-\frac{1}{2})$ be taken together $= C_{-\frac{1}{2}}$, so that

$$C_{-\frac{1}{2}} = -\tfrac{1}{24}f'\,(a-\tfrac{1}{2}) + \tfrac{17}{5760}f'''\,(a-\tfrac{1}{2}) - \tfrac{367}{967680}f^{\mathrm{v}}\,(a-\tfrac{1}{2}) \ldots$$

then $C_{-\frac{1}{2}}$ becomes a constant for every later terminating limit. If $C_{-\frac{1}{2}}$ take the place of the term ${}'f(a-\frac{1}{2})$ in the series ${}'f$, then the first term of the series is

$${}'f(a+\tfrac{1}{2}) = C_{-\frac{1}{2}} + f\,a$$

instead of the above.

To join on the second series, any quantity whatever may be taken in the place of ${}''f(a-1)$, with which is derived ${}''f\,a = {}''f(a-1)$

$+ C_{-\frac{1}{2}}$; $C_{-\frac{1}{2}}$ having taken the place of $'f(a - \frac{1}{2})$ in the series $'f$. In the first place, the value of the double integral up to $n = -\frac{1}{2}$, the beginning of the series, must be investigated. And for this purpose the quantities $''f, f, f''$, &c., must be interpolated for the argument $a - \frac{1}{2}\omega$, treating them as functions of $a - 1\omega$, and a. Let it be observed here, that, $''f$ being the second summed series, $'f$ are the first, and f the second differences of it; and that $f', f'', f''', f^{\text{IV}}$, &c., denote the first, second, third, fourth, &c., differences of the principal function f.

Professor ENCKE's form of interpolation gives, in the several series, the values,

$$''f(a - \tfrac{1}{2}) - \tfrac{1}{8}f(a - \tfrac{1}{2}) + \tfrac{3}{128}f''(a - \tfrac{1}{2}) - \tfrac{5}{1024}f^{\text{IV}}(a - \tfrac{1}{2}) \ldots .$$
$$f(a - \tfrac{1}{2}) - \tfrac{1}{8}f''(a - \tfrac{1}{2}) + \tfrac{3}{128}f^{\text{IV}}(a - \tfrac{1}{2})$$
$$f''(a - \tfrac{1}{2}) - \tfrac{1}{8}f^{\text{IV}}(a - \tfrac{1}{2})$$
$$f^{\text{IV}}(a - \tfrac{1}{2})$$

in which $''f(a - \frac{1}{2}) = \frac{1}{2}''f(a-1) + \frac{1}{2}''f a$ and $f(a - \frac{1}{2} = \frac{1}{2}f(a-1) + \frac{1}{2}f a$; in the same way all the other functions are arithmetical means. The numerical coefficients (as will be seen below) in the general expression of the double integral are for the first series 1, for the second series $+\frac{1}{12}$, for the third series $-\frac{1}{240}$, for the fourth series $+\frac{31}{60480}$; multiplying the respective series by these coefficients, and summing up the terms, gives for the value of the integral $n = \frac{1}{2}$,

$$''f(a - \tfrac{1}{2}) - \tfrac{1}{24}f(a - \tfrac{1}{2}) + \tfrac{17}{1920}f''(a - \tfrac{1}{2}) - \tfrac{367}{193536}f^{\text{IV}}(a - \tfrac{1}{2}) \ldots$$

Since $''f(a-1)$ is wholly arbitrary, it can be put $= 0$; by this means

$$''f(a - \tfrac{1}{2}) = \tfrac{1}{2}''f(a-1) + \tfrac{1}{2}''f a,$$

or, from the above value of $''f a$,

$$= ''f(a - 1) + \tfrac{1}{2}'f(a - \tfrac{1}{2}),$$

becomes $= \frac{1}{2}C_{-\frac{1}{2}}$, or according to the above value of $C_{-\frac{1}{2}}$,

$$''f(a-\tfrac{1}{2}) = -\tfrac{1}{48}f'(a-\tfrac{1}{2}) + \tfrac{17}{12520}f'''(a-\tfrac{1}{2}) - \tfrac{367}{1535360}f^{\text{v}}(a-\tfrac{1}{2})\ldots$$

If this value is substituted in the integral, and it is reduced by means of the following simple equations, expressing the relations of interpolated differences,

$$\begin{aligned}
f\ (a-\tfrac{1}{2}) &= fa - \tfrac{1}{2}f'(a-\tfrac{1}{2}) \\
f''(a-\tfrac{1}{2}) &= \tfrac{1}{2}f''(a-1) + \tfrac{1}{2}f''a \\
f'''(a-\tfrac{1}{2}) &= f''a - f''(a-1) \\
f^{\text{IV}}(a-\tfrac{1}{2}) &= \tfrac{1}{2}f^{\text{IV}}(a-1) - \tfrac{1}{2}f^{\text{IV}}a \\
f^{\text{V}}(a-\tfrac{1}{2}) &= f^{\text{IV}}a - f^{\text{IV}}(a-1)
\end{aligned}$$

then the value of the integral for $n = -\frac{1}{2}$ may be written as follows:

$$-\tfrac{1}{24}fa + \tfrac{17}{5760}\{2f''a + f''(a-1)\} - \tfrac{367}{967680}\{3f^{\text{IV}}a + 2f^{\text{IV}}(a-1)\}\ldots$$

This quantity must be subtracted throughout by reason of the first limit being taken $= -\frac{1}{2}$. If therefore we put in the second summed series, in the place of $''f(a-1)$, this quantity with opposite signs, or

$$C'_{-\frac{1}{2}} = +\tfrac{1}{24}fa - \tfrac{17}{5760}\{2f''a + f''(a-1)\} + \tfrac{367}{967680}\{3f^{\text{IV}}a + 2f^{\text{IV}}(a-1)\}\ldots.$$

this subtraction will be made once for all.

With these substitutions the complete scheme for $n = -\frac{1}{2}$ taken as the first limit is,

Argument.	f Principal Function.	$'f$ I. Summed Series.	$''f$ II. Summed Series.
$a - 1\,\omega$	$f(a-1)$	$C_{-\frac{1}{2}}$	$C'_{-\frac{1}{2}}$
a	fa	$'f(a+\frac{1}{2})$	$''fa$
$a + 1\,\omega$	$f(a+1)$	$'f(a+\frac{3}{2})$	$''f(a+1)$
$a + 2\,\omega$	$f(a+2)$	$'f(a+\frac{5}{2})$	$''f(a+2)$
$a + 3\,\omega$	$f(a+3)$		$''f(a+3)$
	&c.	&c.	&c.

The second term of the second series being now

$$''f\,a = C'_{-\frac{1}{2}} + C_{-\frac{1}{2}}.$$

For every subsequent limit, $a + (i + n'')\,\omega$, the expression is

$$\iint f(a + n\,\omega)\,d\,n^2 = ''f(a + i + n'') + \tfrac{1}{12} f(a + i + n'') \\ - \tfrac{1}{240} f''(a + i + n'') + \tfrac{31}{60480} f^{\text{IV}}(a + i + n'') \ldots\ldots$$

So that for every n'' the several functions must be interpolated as if the values actually occurring in the summed series $''f$ were functions of $a + i\,\omega$.

In the special case of $n'' = \frac{1}{2}$ the interpolation takes place into the middle of the interval between the terms, as shown in the above example of $n = -\frac{1}{2}$.

The integral for this value of n'' is,

$$\iint_{-\frac{1}{2}}^{i+\frac{1}{2}} f(a + n\,\omega)\,d\,n^2 = ''f(a + i + \tfrac{1}{2}) - \tfrac{1}{24} f(a + i + \tfrac{1}{2}) \\ + \tfrac{17}{1920} f''(a + i + \tfrac{1}{2}) - \tfrac{367}{193536} f^{\text{IV}}(a + i + \tfrac{1}{2}) \ldots$$

in which by $''f(a + i + \frac{1}{2})$ is understood the arithmetical mean between $''f(a + i)$ and $''f(a + i + 1)$, and so with all the other functions f, f'', f^{IV}.

It need be mentioned merely, that, in both cases, the generally indispensable factor ω^2 for the double integral is considered as comprised in the functions f, so that, to derive the first integral from the same table of functions, it is necessary, on this account, to divide by ω.

In order to illustrate by an example the formation of the series at the beginning, I have introduced, from Professor Encke's treatise, the following table, of which the table at the end of the previous computation is the continuation, the intervening part of the series being omitted.

0^h P. M. T.	Argument	$42^2 \frac{d\mu}{dt}$	$'f$	$''f$
		$''$	$''$	$''$
1809, Sept. 17	$a - 3\,\omega$	$+6.57219$	. . . .	. . . . . .
Oct. 29	$a - 2\,\omega$	$+5.82592$	. . . .	. . . . . .
Dec. 10	$a - \omega$	$+5.15193$	$+\ 0.02489$	$+\ 0.18914$
1810, Jan. 21	a	$+4.55359$	$+\ 4.57848$	$+\ 0.21403$
March 4	$a + \omega$	$+4.01802$	$+\ 8.59650$	$+\ 4.79251$
April 15	$a + 2\,\omega$	$+3.53914$	$+12.13564$	$+\ 13.38901$
May 27	$a + 3\,\omega$	$+3.10646$	$+15.24210$	$+\ 25.52465$
July 8	$a + 4\,\omega$	$+2.71163$	$+17.95373$	$+\ 40.76675$
Aug. 19	$a + 5\,\omega$	$+2.34609$	$+20.29982$	$+\ 58.72048$
Sept. 30	$a + 6\,\omega$	$+2.00418$	$+22.30400$	$+\ 79.02030$
Nov. 11	$a + 7\,\omega$	$+1.68040$	$+23.98440$	$+\ 101.32430$
Dec. 23	$a + 8\,\omega$	$+1.36911$	$+25.35351$	$+\ 125.30870$
1811, Feb. 3	$a + 9\,\omega$	$+1.06719$	$+26.42070$	$+\ 150.66221$
March 17	$a + 10\,\omega$	$+0.77184$	$+27.19254$	$+\ 177.08291$
April 28	$a + 11\,\omega$	$+0.48183$		$+\ 204.27545$
. . . .	. . . . .	. . .	. . . .	. . . . . .
1836, May 22	$a + 229\,\omega$	-0.7115		-4802.3339
July 3	$a + 230\,\omega$	-1.3811	-19.5017	-4821.8356
Aug. 14	$a + 231\,\omega$	-1.8398	-20.8828	-4842.7184
Sept. 25	$a + 232\,\omega$	-2.0960	-22.7226	-4865.4410
Nov. 6	$a + 233\,\omega$	-2.1546	-24.8186	-4890.2596
Dec. 18	$a + 234\,\omega$	-2.0412	-26.9732	-4917.2328

The integral
$$\varDelta M = \iint . \frac{d\mu}{dt} . dt^2$$
will serve for an example of the double integration, in which μ is the mean daily sidereal motion of Vesta, and $\frac{d\mu}{dt}$ is the differential coefficient, relative to the time, of the perturbation of the μ of Vesta caused by the direct attraction of Jupiter.

By means of the first integral $\int\frac{d\mu}{dt}dt$, is obtained the true μ affected by the perturbations. The second gives the mean anomaly resulting from the perturbations of μ. The unit in $\frac{d\mu}{dt}$ is the mean day, and the interval of time has been taken equal to 42 days. The beginning of both integrals falls upon January 0, 1810. In order to obtain $\varDelta M$ immediately

$$\omega^2 . \frac{d\mu}{dt} = 1764 \frac{d\mu}{dt}$$

has been put as the function f, and the values of $f(a + n\omega)$ have been computed for

$$a - 3\omega,\ a - 2\omega,\ a,\ a + \omega,\ a + 2\omega,\ \&c. \ldots\ldots a + 233\omega,\ a + 234\omega,$$

in which $a =$ January 0, 1810, $+ \frac{1}{2}\omega =$ January 21, 1810.

It will be observed that the constants $C_{-\frac{1}{2}}$, $C'_{-\frac{1}{2}}$, have taken the place in the table of $'f(a - \frac{1}{2})$ and $''f(a - 1)$; in order to obtain these constants we have

$$\begin{aligned} f' \ (a - \tfrac{1}{2}) &= - 0.59834 \\ f'' \ (a - 1) &= + 0.07565 \\ f'' \ a \qquad &= + 0.06277 \\ f''' (a - \tfrac{1}{2}) &= - 0.01288 \end{aligned}$$

It is not necessary to go farther back than f''', because the higher differences are insensible, and the terms are wanting in the beginning of the series to obtain them, except when two computations made from different elements meet in this place. By means of the above values,

$$\begin{aligned} C_{-\frac{1}{2}} &= + 0.02493 - 0.00004 = + 0.02489 \\ C'_{-\frac{1}{2}} &= + 0.18973 - 0.00059 = + 0.18914. \end{aligned}$$

The argument for January 13, 1811, $= a + (8 + \frac{1}{2}\omega)$, whence

$$\begin{aligned} ''f \ (a + \tfrac{17}{2}) &= + 137.98545 \,.\, 5 \\ - \tfrac{1}{24} f \ (a + \tfrac{17}{2}) &= - \quad 0.05075 \,.\, 6 \\ + \tfrac{17}{1920} f'' (a + \tfrac{17}{2}) &= + \quad \underline{0.00007 \,.\, 1} \end{aligned}$$

and $\qquad \varDelta M = + 137.93477$ is the amount of $\varDelta M$ from January 0, 1810, to January 13, 1811.

Again, the argument for July 3, 1836, is $(a + 230\omega)$, and

$$\begin{aligned} ''f \ (a + 230) &= - 4821.8356 \\ + \tfrac{1}{12} f \ (a + 230) &= - \quad 0.1150 \,.\, 9 \\ - \tfrac{1}{240} f'' (a + 230) &= + \quad \underline{0.0008 \,.\, 8} \end{aligned}$$

whence $\qquad \varDelta M = - 4821.9498$ is the double integral of $\varDelta M$ from January 0, 1810, to July 3, 1836.

We may employ the same functions to furnish an example of the single integral, remembering, as before mentioned, that when a single integral is derived from a table of functions comprising the factor ω^2, it must be divided by ω.

For the first argument we have

$$
\begin{array}{rrl}
{}'f & (a + \tfrac{17}{2}) = + & 25.35351 \\
+ \tfrac{1}{24} \; f' & (a + \tfrac{17}{2}) = - & 0.01258 \\
- \tfrac{17}{5760} f''' & (a + \tfrac{17}{2}) = + & \underline{0.00001} \\
 & 42 \, \varDelta \mu = + & 25.34094 \\
 & \varDelta \mu = + & 0.603356
\end{array}
$$

from January 0, 1810, to January 13, 1811.

And for the argument $(a + 232\,\omega)$ we have

$$
\begin{array}{rrl}
{}'f & (a + 232\,\omega) = - & 23.7706 \\
- \tfrac{1}{12} \, f' & (a + 232\,\omega) = - & 0.0131 \,.\, 2 \\
+ \tfrac{11}{720} f''' & (a + 232\,\omega) = - & \underline{0.0001} \\
 & 42 \, \varDelta \mu = - & 23.7838 \,.\, 2 \\
 & \varDelta \mu = & 0.5662 \,.\, 8
\end{array}
$$

from January 0, 1810, to September 25, 1836.

This being the first time that the single integral

$$\int_{-\frac{1}{2}}^{i} f(a + n\,\omega)\, dn,$$

in which n is a whole number, has been introduced, it may be well to notice that the numerical coefficients of the different series are changed from $+ 1$, $+ \frac{1}{24}$, $- \frac{17}{5760}$, to $+ 1$, $- \frac{1}{12}$, $+ \frac{11}{720}$.

Should we desire to take immediately from the table the first and second integrals for any other time, as, for example, for February 10, 1811, or for the argument $a + (9 + \frac{1}{6})\,\omega$, we must interpolate for $42 \, \varDelta \mu$ as follows: —

$$
\begin{aligned}
{}'f\ (a+9+\tfrac{1}{6}) &= {}'f\ (a+9+\tfrac{1}{2}-\tfrac{1}{3}) = +\ 26''.09809 \\
f'\ (a+9+\tfrac{1}{6}) &= f'\ (a+9+\tfrac{1}{2}-\tfrac{1}{3}) = -\ \ \ 0.29740 \\
f'''(a+9+\tfrac{1}{6}) &= f'''(a+9+\tfrac{1}{2}-\tfrac{1}{3}) = -\ \ \ 0.00175;
\end{aligned}
$$

and for $\varDelta M$,

$$
\begin{aligned}
{}''f\ (a+9+\tfrac{1}{6}) &= +\ 154.99961 \\
f\ (a+9+\tfrac{1}{6}) &= \qquad 1.01754 \\
f''(a+9+\tfrac{1}{6}) &= \qquad 0.00636,
\end{aligned}
$$

from which we should obtain

$$
\begin{aligned}
42\,\varDelta\mu &= +\ 26.09809 - 0.01239\,.\,2 + 0.00000\,.\,5 \\
&= +\ 26.08570 \\
\varDelta\mu &= +\ 0.621088 \text{ from Jan. 0, 1810, to Feb. 10, 1811;}
\end{aligned}
$$

and
$$
\begin{aligned}
\varDelta M &= +\ 154.99961 + 0.08479\,.\,5 - 0.00002\,.\,6 \\
&= +\ 155.08438 \text{ from Jan. 0, 1810, to Feb. 10, 1811,}
\end{aligned}
$$

agreeing with the interpolation from the previous values.

When the amount of the remaining perturbations is computed, the actual value of μ for January 0, 1810, is lastly to be added to $\varDelta\mu$, and so also the value which M would have without the perturbations at each time is to be added to $\varDelta M$. The first sum may be denoted by μ_0, and the form of the last will be $M_0 + (n + \frac{1}{2})\,\omega\,\mu_0$. It would only be an unnecessary addition to the calculation to put in the place of the constant $C_{-\frac{1}{2}}$ the value $C_{-\frac{1}{2}} + 42\,\mu_0$, and in the place of $C'_{-\frac{1}{2}}$ the value $C'_{-\frac{1}{2}} + M_0 - \frac{1}{2}\,\omega\,\mu_0$, for the purpose of obtaining the whole result immediately from the table.

Finally, the connection of the two parts of the above table may be shown by computing the value of the formula

$$M = L - \pi + \mu\,(t - T)$$

according to the precept.

$$L_0 = 105^\circ\ 53'\ 15''.63$$

$$\iint_{t' = \text{Jan. } 0, 1810.}^{t'' = \text{June } 12, 1836.} \frac{d\mu}{dt} \cdot dt^2 = -1\ 20\ 12.03$$

$$\int_{t' = \text{Jan. } 0, 1810.}^{t'' = \text{June } 12, 1836.} \frac{dL}{dt} \cdot dt = -0\ 17\ 51.62$$

$$\mu(t - T) = 105\ \ 5\ 46.21$$

$$L = 209\ 20\ 58.2 + 360^\circ = 569^\circ\ 20'\ 58''.2$$

$$\pi_0 = 249\ 48\ 26.9$$

$$\int_{t' = \text{Jan. } 0, 1810.}^{t'' = \text{June } 12, 1836.} \frac{d\pi}{dt} \cdot dt = \ \ 0\ 19\ 29.45$$

$$\pi = 250\ \ 7\ 56.4 \qquad\qquad 250^\circ\ \ 7'\ 56''.4$$

$$M = L - \pi = 319\ 13\ \ 1.8$$

And by the elements answering to the date June 12, 1836.

$$M = 319\ 13\ \ 1.8$$

$$\pi = 250\ \ 7\ 56.4$$

$$L = M + \pi = 209\ 20\ 58.2$$

ON THE DEPENDENCE OF NAPIER'S RULES.

By Rev. Anthony Vallas, Phil. Dr., New Orleans, La.

The five parts of a right-angled spherical triangle being placed on the circumference of a circle, Napier's Rules are as follows: —

Rule I. *The sine of the middle part equals the product of the cosines of the opposite parts.*

Rule II. *The sine of the middle part is equal to the product of the tangents of the adjacent parts.*

It must be remembered that, instead of the hypothenuse and the two acute angles, their complements are used, the right angle not being counted among the parts.

That the second of these rules may be deduced from the first has been shown by Mr. Safford, in No. I. Vol. I. We are going to show the same in the following way. Apply the first rule to the three parts α, α', ω', and then to α, α', ω. Eliminate one of the common parts, for instance α, and the result, properly transformed, will give the second rule. From

$$\sin\alpha = \cos\alpha' \cos\omega', \qquad \sin\alpha' = \cos\alpha \cos\omega,$$

we get
$$\sin^2\alpha' = (1 - \cos^2\alpha' \cos^2\omega') \cos^2\omega ;$$

hence
$$\frac{1}{\cos^2\omega} = \frac{1}{\sin^2\omega} - \cot^2\alpha' \cos^2\omega',$$

$$1 + \tan^2\omega = 1 + \cot^2\alpha' - \cot^2\alpha' \cos^2\omega',$$

from which follows

$$\sin^2\omega' = \tan^2\alpha' \tan^2\omega \qquad \therefore \sin\omega' = \tan\alpha' \tan\omega ;$$

or Napier's Rule II.

In order to find the second rule from the first, take the three adjacent parts α, μ, α', and μ, α', ω', and eliminate μ from the two equations

$$\sin\mu = \tan\alpha \tan\alpha', \qquad \sin\alpha' = \tan\mu \tan\omega'.$$

Since
$$\tan\mu = \frac{\sin\mu}{\sqrt{1 - \sin^2\mu}},$$

we get
$$\sin\alpha' = \frac{\tan\alpha \tan\alpha' \tan\omega'}{\sqrt{1 - \tan^2\alpha \tan^2\alpha'}}; \text{ or}$$

$$1 - \tan^2\alpha \tan^2\alpha' = \frac{\tan^2\alpha \tan^2\omega'}{\cos^2\alpha'}.$$

Replacing the tangents by their equivalents in cosines, we have

$$\frac{1}{\cos^2\alpha \cos^2\alpha' \cos^2\omega'} - \frac{1}{\cos^2\alpha' \cos^2\omega'} - \frac{1}{\cos^2\alpha} = 0; \text{ or}$$

$$1 - \cos^2\alpha - \cos^2\alpha' \cos^2\omega' = 0; \text{ or}$$

$$\sin^2\alpha = \cos^2\alpha' \cos^2\omega' ; \qquad \therefore \sin\alpha = \cos\alpha' \cos\omega',$$

which is Napier's Rule I.

ON A TRANSFORMATION OF THE DERIVATIVE OF ANY POWER OF A VARIABLE.

By Rev. GEORGE CLINTON WHITLOCK, LL.D.,
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THEOREM. — *Any power of a variable, regarded as a factor of any power of the same, may be carried from under the sign of derivation, by adding algebraically the product formed on multiplying its exponent into the variable having an exponent less by unity than that of the undecomposed power ;*

or
$$D(x^n) = x^r D(x^{n-r}) + r\, x^{n-1},$$

whatever n *and* r *may be, plus or minus, whole or fractional, or imaginary.*

DEMONSTRATION. — From the laws of exponents in general, whatever n may be, plus or minus, whole or fractional, or imaginary, we have

$$x^n = x\, x^{n-1};$$

$$\therefore\ (x+h)^n = (x+h)\,(x+h)^{n-1},$$

by changing x into $(x+h)$; consequently, subtracting member by member and separating the factors x, h,

$$(x+h)^n - x^n = x\,[(x+h)^{n-1} - x^{n-1}] + h\,(x+h)^{n-1},$$

and, dividing by h,

$$\frac{(x+h)^n - x^n}{h} = x\,\frac{(x+h^{n-1} - x^{n-1}}{h} + (x+h)^{n-1};$$

therefore, passing to the *limit*, that is, taking the values of the terms when h reduces to zero,

$$\lim \frac{(x+h)^n - x^n}{h} = x \lim \frac{(x+h)^{n-1} - x^{n-1}}{h} + x^{n-1}, \text{ or}$$

(1)
$$D(x^n) = x\,D(x^{n-1}) + x^{n-1};\ \text{i. e.}$$

the derivative of x^n is equal to x times the derivative of x^{n-1} plus

x^{n-1}; or a factor x of any power x^n may be carried from under the sign D, of derivation, by adding x^{n-1}.

Therefore by (1)

$$D(x^{n-1}) = x\,D(x^{n-2}) + x^{n-2},$$

and by replacing $D(x^{n-1})$ by this value, (1) becomes

$$D(x^n) = x^2\,D(x^{n-2}) + 2\,x^{n-1};$$

and in like manner this form may be changed into

$$D(x^n) = x^3\,D(x^{n-3}) + 3\,x^{n-1};$$

and in general, the same operation being repeated r times, we have

(2) $$D(x^n) = x^r\,D(x^{n-r}) + r\,x^{n-1};\text{ i. e.}$$

any number, r, of factors, x, may be carried from under the sign D by adding $r\,x^{n-1}$.

From (2), dividing by x^r, we have

$$x^{-r}\,D(x^n) = D(x^{n-r}) + r\,x^{n-r-1},\text{ or}$$

$$D(x^{n-r}) = x^{-r}\,D(x^n) - r\,x^{n-r-1},\text{ or}$$

(3) $$D(x^m) = x^{-r}\,D(x^{m+r}) - r\,x^{m-1},$$

putting $n - r = m$, which may be any number whatever, since n is any number; hence (3), with proper observance of sign, for the separation of reciprocal factors, $x^{-r} = \frac{1}{x^r}$, we have the same rule as above, (1).

Again (2, 3), ϱ and r being any whole numbers, plus or minus,

(4) $$D(x^n) = x^{\varrho r}\,D(x^{n-\varrho r}) + \varrho\,r\,x^{n-1};\text{ i. e.}$$

a factor x^r being separated, a power of this factor, $(x^r)^\varrho = x^{\varrho r}$, will be separated by multiplying r by ϱ.

It follows that

$$(5) \qquad D(x^n) = x^{\frac{r}{\nu}} D\left(x^{n-\frac{r}{\nu}}\right) + \frac{r}{\nu} x^{n-1},$$

since by the last, multiplying $\frac{r}{\nu}$ by ν, we have

$$D(x^n) = x^r D(x^{n-r}) + r x^{n-1},$$

which, as necessary and sufficient, accords with (2). Therefore (2), (3), (5), the theorem is proved, since whatever is found for real exponents must be extended to those that are imaginary.

Scholium. — The deduction of (3) and (5) from (2) is in accordance with the general theory of exponents, as indicating, whether plus or minus, whole or fractional, or imaginary, the number of times the quantity to which they are affixed, must be taken as a factor, or so regarded, in executing any operation. Thus x^r is x taken r times as a factor, and theory requires that x^{-r} be regarded as x taken $-r$ times as factor, and so $x^{\frac{1}{\nu}}$ is x taken $\frac{1}{\nu}$ times as factor, and $x^{\frac{r}{\nu}} = \left(x^{\frac{1}{\nu}}\right)^r$ is $x^{\frac{1}{\nu}}$ taken r times as factor, or x taken $r \cdot \frac{1}{\nu}$ or $\frac{r}{\nu}$ times as factor. It is true that here, as elsewhere, we have to extend the signification of the phrase, "*number of times*," that is, "*number*" must be regarded, not only as *continuous* (of any magnitude between integers), but as plus, or minus, or imaginary. And it is by this extension, *precisely*, that Analysis has its *power*, or better, an *existence*.

Corollary. — Whatever n may be, from the theorem we have

$$D(x^n) = x^n D(x^0) + n x^{n-1};$$

but $\quad D(x^0) = \lim \frac{(x+h)^0 - x^0}{h} = \lim \frac{1-1}{h} = \lim \frac{o}{h} = \lim 0 = 0;$

$$\therefore D(x^n) = n x^{n-1}; \text{ i. e.}$$

the derivative of any power of a variable is found by diminishing the exponent by unity and multiplying by the original exponent.

EXPOSITION OF THE PROCESS OF MATHEMATICAL DEVELOPMENT.

By JOHN PATERSON, A. M., Albany, N. Y.

1. THE measure of an effect is adopted as the measure of its cause; and in mathematical investigations, the effect is the measure of the force that produces it. A constant force produces equal effects in equal times; but when the force is variable, the results in equal successive times are unequal. Beginning with zero, if a force increase by a regular law during a given time, and then remain constant with its ultimate value during an equal time, the effect in the latter interval of time will be some determinate number m of times its effect in the former interval.

The immediate product of a force may be itself a force, and in its turn produce a third force, and so on for several descending ranks or orders of generated and generating forces, decreasing in intension but increasing in extension, until finally arriving at the order zero, which will be the phenomenal effect, the measure of which in space and time is the *base line* of reference for the determination of the genetical operations of the subordinated system of forces under consideration. For instance, the immediate effect of the force of gravitation in the extent of space for which it may be regarded as a constant or uniform activity, is the product or genesis of a uniform velocity; and this (unresisted) velocity causes the body in which it is developed to describe a distance which increases as the square of the time, which distance is the phenomenal effect, and its measure for the first unit of time occupied in the genetical process is the unit measure of reference.

2. Let φ^n be a constant force of the nth order; it generates force φ^{n-1} of the $(n - 1)$th order uniformly in time. As φ^{n-1} increases

uniformly, as in the simple ratio of the time, it generates force φ^{n-2} of the $(n - 2)$th order with a uniformly accelerated energy, or in duplicate ratio of the time; so that the increase of the latter force increases uniformly, or the force itself is duplicately accelerated. In its turn φ^{n-2} generates force φ^{n-3} of the $(n - 3)$th order with a duplicately accelerated energy, so that the increase of the increase of the latter force increases uniformly, or the force itself is triplicately accelerated. This process of augmented acceleration continues through successive descending orders, until arriving at the force $\varphi^{n-n} = \varphi^0$ of the order zero, the phenomenal effect, which is measurable in space and time, and serves to determine the measures of the several generating forces.

For greater simplicity, the phenomenal effect produced or generated in a unit of time 1_t may be adopted as the unit of effect in the system of operations. Beginning at zero with the genetical operations, the measure of the phenomenal effect in the first unit $1'_t$ of time is denoted by 1_l the linear unit, this being the distance that the unit of mass 1_μ is assumed to be carried in that time by the force $\varphi^{n-n+1} = \varphi'$ of the first order, as a general and simple representative of the unit measure of an operation. This definition requires that unity, expressed in the coefficient 1, shall be the measure of the force φ', and indeed also of each member of the descending series φ^n, φ^{n-1}, φ^{n-2}, ... for the first unit $1'_t$ of time occupied in the genetical process; a requirement that is obviously legitimately based upon the logical admeasurement of causes by their effects, and is indispensable for the determination of the effects that will arise in the succeeding units of time. All the forces from φ^{n-1} to φ' inclusive are variable; each increases from 0 to some determinate magnitude, $m\,\varphi^{n'}$ in the first unit of time $1'_t$, and then remains a constant or uniform generator during the second unit of time $1''_t$ with the ultimate value it has acquired at the termination

of the first unit of time. The ratio of the effect of each force thus conditioned, during the second interval $1''_t$ of time, to its effect during the equal interval $1'_t$ of time occupied in its own genesis or growth, is the very relation to be established.

3. The constant force φ^n generates equal increments of force φ^{n-1} in successive equal intervals of time $1'_t$, $1''_t$, $1'''_t$, etc. The force φ^{n-1} therefore increases uniformly from 0 during the first unit of time $1'_t$; and if then remaining constant with its acquired value during the second unit of time $1''_t$, it will generate some definite number m of times the amount of force φ^{n-2} that was generated by it during the equal time $1'_t$ of its own increase or growth; that is, if φ^{n-1}, with a uniformly increasing energy, has generated $1\,\varphi^{n-2}$ in $1'_t$, then with its ultimate energy remaining constant, it will generate $m\,\varphi^{n-2}$ during $1''_t$. Then φ^{n-1} has increased from 0 to the value $m\,\varphi^{n-1}$ during the time $1'_t$; and during this time $1'_t$ of its increase, it has generated the mth part as much as will be its effect as a constant generator during the equal time $1''_t$.

4. As φ^{n-1} increases uniformly, its value at the expiration of the first semi-unit $\frac{1}{2}'.1'_t$ of time will be $\frac{m}{2}\varphi^{n-1}$; with this value remaining constant, it would generate $\frac{m}{2}\varphi^{n-2}$ in a unit of time, and consequently it does generate $\frac{m}{4}\varphi^{n-2}$ in the second semi-unit $\frac{1}{2}''.1'_t$ of time, the mth part of which amount is $\frac{1}{4}\varphi^{n-2}$ the effect of $\frac{m}{2}\varphi^{n-1}$ during the time $\frac{1}{2}'.1'_t$ of its own genesis. During the second semi-unit $\frac{1}{2}''.1'_t$ of the first unit $1'_t$ of time, a second equal increment $\frac{m}{2}\varphi^{n-1}$ is generated by the primitive generator $1\,\varphi^n$, and this increment likewise generates $\frac{1}{4}\varphi^{n-2}$ during the time of its own genesis. The sum of these effects, estimated in and for the first unit of time, is conditioned to be unity; hence the equation $\left(\frac{1}{4}+\frac{m}{4}+\frac{1}{4}\right)\varphi^{n-2}=1\,\varphi^{n-2}$.

In the equations

$$\tfrac{1}{2}'.1'_t + \tfrac{1}{2}''.1'_t = 1'_t;$$
$$\tfrac{1}{2}\varphi_1^{n-1} + \tfrac{1}{2}\varphi_2^{n-1} = 1\,\varphi^{n-1},$$
$$\tfrac{m}{2}\varphi_1^{n-1} + \tfrac{m}{2}\varphi_2^{n-1} = m\,\varphi^{n-1},$$
$$\left.\begin{array}{l}\tfrac{1}{4}\varphi_1^{n-2} + \tfrac{m}{4}\varphi^{n-2} \\ \qquad + \tfrac{1}{4}\varphi_2^{n-2}\end{array}\right\} = 1\,\varphi^{n-2},$$

the terms of the left-hand member of the first are the constant equivalents of the corresponding varying terms of the second, and those of the third exhibit the effects generated by their correspondents of the second, the measures of these effects being confined to what shall abide throughout in the first unit of time. The third equation gives $m = 2$, which value converts the immediately preceding one into $\tfrac{2}{2}\varphi^{n-1} + \tfrac{2}{2}\varphi^{n-1} = 2\,\varphi^{n-1}$, and proves that when a force increases uniformly from 0 during a given time, and then remains constant during an equal time, its effect during the second interval will be twice that during the first interval. Thus the *successive* and *synchronous* unit measures of a force of the second rank (from its constant generator) are mutually convertible by the factor 2.

5. During the time $1'_t$, the entire force φ^{n-2} increases from 0 in duplicate ratio; and consequently its effect in the time $1''_t$, with its acquired or ultimate value in $1'_t$ remaining constant, will be some number m' of times its effect in the time $1'_t$; this ultimate value of φ^{n-2} is made up of the sum of the ultimate values of the partial forces of which $m'\,\varphi^{n-2}$ is constituted, and therefore the transition from the first to the second of the following equations must be shown.

$$\tfrac{1}{2}'.1'_t + \tfrac{1}{2}''.1'_t = 1'_t;$$
$$\left.\begin{array}{l}\tfrac{1}{4}\varphi_1^{n-2} + \tfrac{2}{4}\varphi^{n-2} \\ \qquad + \tfrac{1}{4}\varphi_2^{n-2}\end{array}\right\} = 1\,\varphi^{n-2},$$
$$\left.\begin{array}{l}\tfrac{m'}{4}\varphi_1^{n-2} + 2\tfrac{m'}{4}\varphi^{n-2} \\ \qquad + \tfrac{m'}{4}\varphi_2^{n-2}\end{array}\right\} = m'\,\varphi^{n-2}.$$

The first and third terms of the left-hand members correspond as constant to variable as in the preceding case, but this remark does not directly apply to the second terms.

The force $\frac{m'}{4}\varphi_1^{n-2}$ would generate $\frac{m'}{4}\varphi^{n-3}$ in a unit of time, and therefore does generate $\frac{m'}{8}\varphi^{n-3}$ in the time $\frac{1}{2}''.1'_t$, the m'th part of which is $\frac{1}{8}\varphi^{n-3}$ the amount it has generated during the time $\frac{1}{2}'.1'_t$ of its own genesis, and is likewise the amount generated by $\frac{m'}{4}\varphi_2^{n-2}$ in the time $\frac{1}{2}''.1'_t$ of its own genesis.

The force $2\frac{m'}{4}\varphi^{n-2}$ would generate $2\frac{m'}{4}\varphi^{n-3}$ in a unit of time, or $2\frac{m'}{8}\varphi^{n-3}$ in a semi-unit of time; but whereas the forces $\frac{m'}{4}\varphi_1^{n-2}$ and $\frac{m'}{4}\varphi_2^{n-2}$ each increase in m'plicate ratio during their genesis, and therefore have the value of their successive effect divided by m' to get that of their synchronous effect, the force $2\frac{m'}{4}\varphi^{n-2}$ increases in duplicate ratio from 0 to this, its ultimate value, during the time $\frac{1}{2}''.1'_t$, being itself generated by the constant $\frac{2}{2}\varphi_1^{n-1}$ in this time, so that the preceding result must be divided by 2 to give $\frac{2}{2}\frac{m'}{8}\varphi^{n-3}$ as the amount generated by $2\frac{m'}{4}\varphi^{n-2}$ during its own genesis in the time $\frac{1}{2}''.1'_t$.

As the result in and for the time $1'_t$ must be unity, the collected effects constitute the equation $\left(\frac{1}{8}+\frac{2}{2}\frac{m'}{8}+\frac{m'}{8}+\frac{1}{8}\right)\varphi^{n-3}=1\,\varphi^{n-3}$, which gives $m'=3$, and converts the last preceding equation into

$$\frac{1}{2}'.1'_t \quad + \frac{1}{2}''.1'_t = 1'_t\,;$$

$$\left.\begin{array}{l}\frac{3}{4}\varphi_1^{n-2}+2\,.\,\frac{3}{4}\varphi^{n-2}\\ \qquad +\frac{3}{4}\varphi_2^{n-2}\end{array}\right\}=3\,\varphi^{n-2}.$$

Thus the successive and synchronous measures of a force of the third rank are mutually convertible by the factor 3.

[To be Continued.]

Mathematical Monthly Notices.

Résumé de Leçons de Géométrie Analytique et de Calcul Infinitésimal, Comprenant sur la Trigonométrie, sur l'Expression des Lieux Géométriques par leurs Équations, sur le Calcul différential et sur les Calcul intégral, l'Exposition des Connaissances nécessaires aux Ingénieurs pour l'Intelligence de la Mécanique rationnelle, de l'Hydraulique et de la Théorie dynamique des Machines. Par J. B. BELANGER, Ingénieur en Chef des Ponts et Chaussées, Professeur de Mécanique à l'Ecole impériale Polytechnique et à l'Ecole Centrale des Arts et Manufactures. Seconde Édition. Paris: Mallet-Bachelier. 1859. pp. 295. Tigr. 104.

To this full title-page but little need be added to give a good idea of the work before us. It is simply intended to supply all the elementary knowledge in Trigonometry, Analytic Geometry, and the Calculus, which the student requires to read the course of rational mechanics as taught in the French schools of engineering. The subjects are clearly treated, and the only peculiarity we notice is, that the subject of Trigonometry is made to depend upon the theory of projections.

Théorie des Fonctions doublement Périodiques et, en particulier des Fonctions Elliptiques. Par M. BRIOT et M. BOGUET. Paris: Mallet-Bachelier. 1859. pp. 342.

The subjects discussed in this book are sufficiently indicated by the title. The method of discussion, which, in fact, involves a completely new "theory of functions," is one, probably, almost *wholly unknown* to American students. The authors have developed and extended the theory of imaginaries named by the illustrious CAUCHY "*les quantités géométriques;*" of which a full discussion may be found in his "*Exercices d'Analyse et de Physique Mathématique*," Tome IV.; and of which some application is made by M. V. PUISEUX in his "*Recherches sur les Fonctions Algébriques*," in LIOUVILLE'S "*Journal de Mathématiques*," Tome XV., p. 365. This view of $\sqrt{-1}$ is founded on the theories of ARGAND, MOUREY, etc. CAUCHY'S "*geometric quantity*," which he denotes by r_θ is algebraically expressed by $x + y\sqrt{-1} = r\,(\cos\theta + \sin\theta \,.\, \sqrt{-1}) = r\, e^{\theta\sqrt{-1}}$, and these quantities may be added, multiplied, etc. just as *coplanar quaternions* would be. The *modulus* r corresponds to the *tensor*, and the *argument* θ to the *angle* of the quaternion. (See p. 133 of the present volume of the Monthly, equation (34).) We have no doubt that this book, especially if read with the two articles we have cited, presents an excellent opportunity to form an acquaintance with the recent progress of the higher branches of analysis among French mathematicians, just as SALMON'S books and BOOLE'S "*Differential Equations*" (heretofore noticed in the Monthly) do for the widely different paths of discovery pursued of late, more especially by British investigators. We scarcely need add, that the *Elliptic Transcendants* appear here in a totally different dress from that given them by their first great expounder, LEGENDRE; or even in the refined analysis of ABEL, although, as with the latter, the inverse function forms the base of the investigations. A second volume is promised, to contain the requisite tables, with practical applications, etc. — B.

Leçons sur les Coördonnées Curvilignes et leurs diverses Applications. Par G. LAMÉ. Paris: Mallet-Bachelier. 1859. pp. 368.

In some of his other books, the author has introduced, to a greater or less extent, the use of these peculiar coördinates invented by himself. If we conceive a family of similar ellipsoids depending on the variable parameter α, such that when α varies we pass from one ellipsoid to the next, etc.; and in the same way a family of one-sheeted hyperboloids depending on a parameter β, and one of two-sheeted hyperboloids depending on γ; then α, β, and γ will be *curvilinear coördinates* of the common intersection-points of these three surfaces. This and similar

systems have been found convenient, if not necessary, in many physical investigations, especially those involving the consideration of *level surfaces*, *isothermal surfaces*, etc. The present volume is designed to explain in their most general form the properties of such systems, with sufficient applications to the "equations of motion," the properties of heat, elasticity, etc., to make their use familiar to the student. No one who wishes to keep himself informed on all valuable advances in recent science should be inattentive to these investigations of Lamé. — B.

Tracts, Mathematical and Physical. By Henry Lord Brougham, LL. D., F. R. S., Member of the National Institute of France, Royal Academy of Naples, Chancellor of the University of Edinburgh. London and Glasgow: Richard Griffin & Co. 1860. 8vo. pp. 304.

The distinguished author's dedication, "To the University of Edinburgh, these tracts, begun while its Pupil, finished when its Head, are inscribed by the Author, in grateful remembrance of benefits conferred of old, and honors of late bestowed," suggest the long period (62 years) within which these tracts were written. Their titles will give the best general idea of their character: General Theorems, chiefly Porisms in the Higher Geometry; Kepler's Problem; Dynamical Principle, — Calculus of Partial Differences, — Problem of Three Bodies; Greek Geometry, Ancient Analysis, Porisms; Paradoxes imputed to the Integral Calculus; Architecture of Cells of Bees; Experiments and Investigations on Light and Colors; Inquiries, Analytical and Experimental, on Light; On Forces of Attraction to Several Centres; Meteoric Stones; Central Forces, and Law of the Universe Analytically Investigated; Attraction of Bodies, or Spherical and Non-spherical Surfaces Analytically treated; Sir Isaac Newton, — Grantham Address; Notes. These Essays possess a peculiar value for the student of mathematics and physics; and even those profoundly versed in the subjects of which they treat will not find them devoid of interest.

Editorial Items.

The following ladies and gentlemen have sent us solutions of the Prize problems in the February number of the Monthly: —

Harriet S. Hazeltine, Worcester, Mass., Probs. I., II.
Amanda M. Bennett, Saline, Washtenaw Co., Mich., Probs. I., II.
T. E. Tower, Amherst College, Probs. I., II., III., IV., V.
D. G. Bingham, Ellicotteville, N. Y., Probs. I., II.
M. K. Bosworth, Marietta College, Ohio, Probs. I., II., V.
Carlos, West Point, N. Y., Prob. I.
James F. Roberson, Indiana University, Bloomington, Probs. II., IV.
George B. Hicks, Cleveland, Ohio, Probs. III., IV., V.
J. Kendrick Upton, New London Institute, N. H., Probs. I., II.
David Trowbridge, Perry City, N. Y., Probs. I., II., III., IV., V.
W. H. Spencer and C. Y. Baldwin, Madison University Grammar School, each Prob. I.
Charles H. Andrews, ——, Prob. I.
H. C. Corey, Exeter, N. H., Probs. I., II.
Asher B. Evans, Madison University, Hamilton, N. Y., Probs. III., IV., V.
Gustavus Frankenstein, Springfield, Ohio, Prob. IV.
Frank N. Devereux, Boston, Mass., Prob. II.

THE

MATHEMATICAL MONTHLY.

Vol. II. . . . JUNE, 1860. . . . No. IX.

PRIZE PROBLEMS FOR STUDENTS.

I. The sum of the radii of the inscribed circle and of the circle touching the hypothenuse and extension of the two legs of a right-triangle is equal to the sum of the said legs.

II. The side of a regular octagon, inscribed in a circle whose radius is unity, is $\sqrt{(2 - \sqrt{2})}$, and its area is $2\sqrt{2}$. Required the proof.

III. The perpendicular to the normal of a parabola at its intersection with the principal axis, meets the curve in two points through which tangents are drawn; find the locus of their common point. — Communicated by Arthur W. Wright, New Haven, Ct.

IV. Required the radii of three equal circles tangent to each other, and cutting off equal areas from a given circle, so that the sum of these areas may be a maximum. — Communicated by G. B. Vose.

V. Two great circles are drawn at random on a sphere. What is the probability that their mutual inclination, taken less than 90°, will be contained between any given limits, as $n°$ and $m°$.

Solutions of these problems must be received by August 1, 1860.

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. VI., Vol. II.

The first Prize is awarded to GEORGE A. OSBORNE, JR., Lawrence Scientific School, Cambridge, Mass.

The second Prize is awarded to DAVID TROWBRIDGE, Perry City, Schuyler Co., N. Y.

The third Prize is awarded to Miss AMANDA M. BENNETT, Saline, Washtenaw Co., Mich.

PRIZE SOLUTION OF PROBLEM I.

By JOHN J. CARTER, Nunda Lit. Institute, N. Y.

Find x from the equation, $\sqrt[5]{(a+x)} + \sqrt[5]{(a-x)} = b$, by quadratics.

Raising the given equation to the fifth power and reducing, it becomes $2a + 5b^3(a^2 - x^2)^{\frac{1}{5}} - 5b(a^2 - x^2)^{\frac{2}{5}} = b^5$;

or
$$(a^2 - x^2)^{\frac{2}{5}} - b^2(a^2 - x^2)^{\frac{1}{5}} = \frac{2a - b^5}{5b}.$$

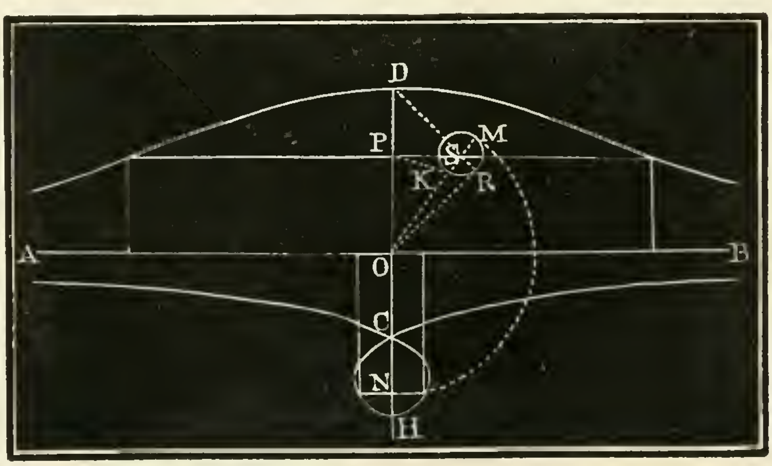
This quadratic gives

$$x = \pm \left\{ a^2 - \left(\frac{b^2}{2} \pm \sqrt{\frac{8a + b^5}{20b}} \right)^5 \right\}^{\frac{1}{2}}$$

PRIZE SOLUTION OF PROBLEM III.

By GEORGE A. OSBORNE, JR, Lawrence Scientific School, Cambridge, Mass.

Inscribe the maximum rectangle between the conchoid and its directrix.



Let AB be the directrix, and C the pole of the conchoid. Denote by a and b, respectively, the distances OD and OC. Then the equation of the curve referred to AB and CD as axis of X and Y, respectively, will be

(1) $$x^2 y^2 = (a^2 - y^2)(b + y)^2.$$

Since the curve is symmetrical with respect to the axis of Y, the inscribed rectangle will evidently be a maximum when xy or $x^2 y^2$ is a maximum.

Differentiating (1) gives

(2) $$D_y (x^2 y^2) = 2(b + y)(a^2 - by - 2y^2)$$

(3) $$D_y^2 (x^2 y^2) = 2(a^2 - b^2 - 6by - 6y^2).$$

Putting $D_y (x^2 y^2) = 0$ gives

(4) $$y + b = 0, \qquad \text{or } a^2 - by - 2y^2 = 0.$$

It is evident that the value $y = -b$ cannot correspond to a maximum, since the rectangle then vanishes.

The roots of (4) are found to be

$$y_1 = -\frac{b}{4} + \sqrt{\left(\frac{b}{4}\right)^2 + \frac{a^2}{2}}, \quad \text{and } y_2 = -\frac{b}{4} - \sqrt{\left(\frac{b}{4}\right)^2 + \frac{a^2}{2}}.$$

Substituting in (3) the value of y^2 from (4) gives

$$D_y^2 (x^2 y^2) = -2(2a^2 + b^2 + 3by).$$

Whatever the relative values of a and b, y_1 is essentially positive, and renders $D_y^2 (x^2 y^2) < 0$.

Hence there will always be a maximum rectangle between the directrix and the *superior* branch of the conchoid. The negative root y_2 will render $D_y^2 (x^2 y^2) < 0$, when $\frac{b}{4} + \sqrt{\left(\frac{b}{4}\right)^2 + \frac{a^2}{2}} < \frac{2a^2 + b^2}{3b}$, which condition is found by reduction to be identical with $b < a$.

Moreover, when $b < a$ the numerical value of y_2 is less than a and greater than b. It therefore lies between CO and HO.

Hence we conclude that when $b < a$, that is to say, when the inferior branch of the conchoid forms a loop CH around the axis of Y, a maximum rectangle may be inscribed between the directrix and the curve, whose lower base is inscribed in this loop; but when

$b > a$, or when this loop does not exist, there is no maximum rectangle for the inferior branch.

To construct these rectangles, draw OR and DR making with OD angles of 45°. Take $RS = \frac{b}{4}$ and describe the circle MRK. Making OP and ON equal respectively to OK and OM determines the position of the required rectangles. It will be readily seen that this is merely a geometrical construction of y_1 and y_2.

Prize Solution of Problem IV.

By George A. Osborne, Jr., Lawrence Scientific School, Cambridge, Mass.

Given a cask containing a gallons of wine. Through a cock at the bottom of the cask wine flows out at the rate of b gallons per minute, and through a hole at the top water flows in at the same rate. Supposing the water, as fast as it flows in, to mingle perfectly with the wine, how long before the quantities of wine and water in the cask will be equal? and how much wine will be left in the cask at the end of t minutes?

Let x denote the number of gallons of wine in the cask at the end of t minutes. Then a being the number of gallons of the mixture, which escapes at the rate of b gallons per minute, the wine escapes at the rate of $\frac{x}{a} b$ gallons per minute.

The quantity of wine which escapes during the time dt is $\frac{x}{a} b\, dt = -dx$ since x varies inversely as t.

From this equation we have by integration $\frac{b}{a} t = -\log x +$ constant. Since $x = a$ when $t = 0$, $\frac{b}{a} t = \log \frac{a}{x}$, or $e^{\frac{bt}{a}} = \frac{a}{x}$, where e is the base of Napier's Logarithms. This gives the quantity of wine in the cask at any time t.

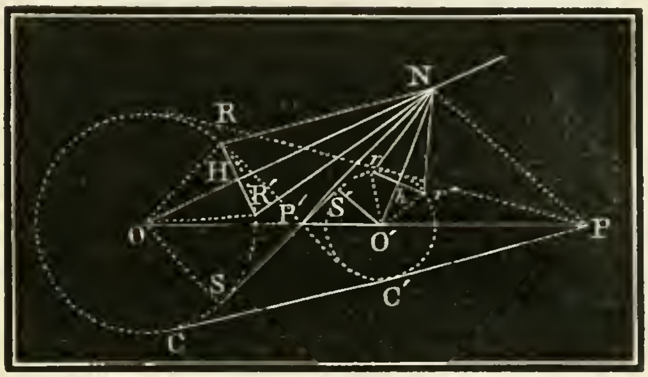
If the quantities of wine and water are equal $x = \frac{a}{2}$ and $t = \frac{a}{b} \log 2$.

Prize Solution of Problem V.

By Isaac H. Turrell, Mount Carmel, Ind.

Two circles being given in a plane, find geometrically the locus of the points from which chords of similar arcs in the two circles will be seen under the same angle, the chords being perpendicular to the lines of vision drawn through the centres of the given circles.

Let O and O' be the centres of the given circles. Draw the common tangents $C\,C'\,P$, $S\,P'\,S'$ cutting $O\,O'$ in P and P'. These points, which are called the external and internal centres of similitude of the two given circles, are evidently the only points on $O\,O'$, from which chords of similar arcs in the two circles will appear under the same angle. Let N be any other point from which the similar chords $R\,R'$, $r\,r'$, which are perpendicular to $N\,H\,O$, $N\,h\,O'$, respectively, will be seen under the equal angles $R\,N\,R'$ $r\,N\,r'$.



Since the triangles $O\,R\,N$, and $O'\,r\,N$ are similar

$$O\,N : O'N :: O\,R : O'r :: O\,P' : O'\,P' :: O\,P : O'\,P.$$

Therefore the line $N\,P'$ bisects the angle $O\,N\,O'$, and $N\,P$ bisects the supplement of $O'\,N\,O$. Therefore $P'\,N\,P$ is a right angle.

Consequently the required locus is a circle described on $P\,P'$ as a diameter.

Cor.—If the two given circles O and O' be considered spheres, the locus of the points will be a sphere described on $P\,P'$ as a diameter.

ON SPHERICAL ANALYSIS.

By George Eastwood, Saxonville, Mass.

[Continued from Page 272.]

Proposition V.

To find the equation of a great circle of the sphere which passes through one or more given points.

1. In Prop. III. we have found the equation of a great circle to be

$$y = \tau\,x + \beta.$$

That this circle may pass through a given point $x'\,y'$, we signify that it does so by saying that when the variable arc x becomes x', then y will become y'. Consequently the equation of the great circle becomes

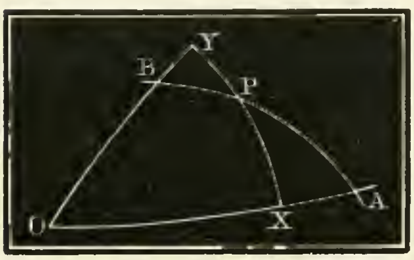
$$y' = \tau\, x' + \beta.$$

Between these two equations, if we eliminate β, we have

$$(23) \qquad y = \tau\, x + y' - \tau\, x', \qquad \text{or } y - y' = \tau\,(x - x'),$$

for the equation of a great circle which passes through a given point whose co-ordinates are $x'\,y'$.

When the given point $x'\,y'$ is situated at 90° from the origin, then $\tan x' = \tan y' = \infty$; and equation (23) is not applicable in this case. We can, however, overcome this apparent difficulty by introducing into our investigation the latitude of the point. For this purpose, let us consider the triangle $Y\,O\,X$, which is formed by the two centres of projection and the origin. By supposition, the great circle $A\,B$ cuts the side $Y\,X$ in the point P, the latitude of which we will designate by λ. Hence, by the theorem of spherical transversals, we have the ratios



$$\sin(90° - \beta) : \sin(\omega - \lambda) = \sin P : \sin B,$$
$$\sin\alpha : \sin\beta = \sin B : \sin A,$$
$$\sin\lambda : \sin(\alpha - 90°) = \sin A : \sin P,$$

which, by composition and expansion, become

$$\cos\beta\,\sin\alpha\,\sin\lambda = -\cos\alpha\,\sin\beta\,\sin(\omega - \lambda).$$

This equation may be expressed in the form

$$\frac{\sin\lambda}{\sin(\omega - \lambda)} \cdot \frac{\sin\alpha}{\cos\alpha} \cdot \frac{\cos\beta}{\sin\beta} = -1;$$

or in this,

$$\frac{\beta}{\alpha} = -\frac{\sin\lambda}{\sin(\omega - \lambda)}.$$

Hence the equation of a great circle which passes through a given point, situated at 90° from the origin, will be truly defined by the equation

(24) $$y = \tau x + \beta,$$

in which τ represents $\frac{\sin \lambda}{\sin(\omega - \lambda)}$, or simply $\tan \lambda$, if the axes be taken rectangular.

Cor.—If any number of great circles have a common point of intersection, situated at 90° from the origin, the coefficient of x will be the same in all their equations.

2. If the proposed circle passes through the given points $x' y'$ and $x'' y''$, we must have by virtue of 1,

(a) $y = \tau x + \beta$, (b) $y' = \tau x' + \beta$, (c) $y'' = \tau x'' + \beta$.

Now $(a) - (b)$ gives

(d) $$y - y' = \tau(x - x'),$$

and $(b) - (c)$ gives

$$y' - y'' = \tau(x' - x''), \qquad \therefore \tau = \frac{y' - y''}{x' - x''}.$$

Substituting this value of τ in (d), we have finally

(25) $$y - y' = \frac{y' - y''}{x' - x''}(x - x').$$

This equation will assume different forms according to the particular position of the given points. For instance, if the point $x'' y''$ be on the axis of x, we have $y'' = 0$;

$$\therefore y - y' = \frac{y'}{x' - x''}(x - x').$$

If it be on the axis of y, then $x'' = 0$;

$$\therefore y - y' = \frac{y' - y''}{x'}(x - x').$$

If it be at the origin, then both x'' and $y'' = 0$;

$$\therefore y = \frac{y}{x'} x.$$

And if it be at 90° from the origin, then by virtue of (24)

$$y - y' = \tau (x - x').$$

If a great circle pass through three given points, the following relation must exist between the co-ordinates of those points;

$$(26) \quad (y' x'' - x' y'') - (y' x''' - x' y''') + (y'' x''' - x'' y''') = 0.$$

[To be continued.]

THE ECONOMY AND SYMMETRY OF THE HONEY-BEES' CELLS.

By Chauncey Wright, Nautical Almanac Office, Cambridge, Mass.

The economical characteristics of the honey-cell have claimed so great attention from mathematicians, that other and even more prominent properties of its structure, though not unnoticed, have received too little attention. Psychologists, accordingly, following the testimony of mathematicians, have treated of the instinct of bees as if economy were not simply the most important, but even the only useful and noticeable feature of the bees' architecture. In reconsidering therefore the geometrical properties of the hive-cell we have chiefly in view the inferences which may be drawn from them in regard to the nature and powers of the bee's instinct.

In this review we shall first recount the reasoning *a priori* by which the structure of the hive-cell is deduced from considerations of rational economy, and then, in the second place, deduce the same results from considerations of symmetry applied to those modifications of the simple nest which are required by simple economy.

It is necessary here to premise a division of economy into two species, which we may denominate rational and sensible, or the

economy of forethought and simple economy; that which forestalls waste, and that which remedies waste or simply saves. The importance of this distinction will be apparent when, in the third place, we consider the two previous lines of argument in their relation to the faculties of instinct.

1. It is a proposition of elementary geometry that of all polygons or rectilinear plane figures with a given number of sides and a given perimeter, that figure contains the greatest area which is regular, or both equilateral and equiangular; and hence, that regular prisms have the greatest solid contents for a given convex surface. Again it is proved that a given perimeter encloses the greatest area in that regular polygon which has the greatest number of sides, so that among plane figures the circle has the greatest area for a given perimeter; and hence, also, the cylinder has the greatest solid contents among prismatic bodies for a given convex surface.

If now we seek among all regular polygons, or prisms, for those which are capable of dividing space into equal and similar parts without interstices, we readily perceive that the angles of such a figure must be aliquot parts of the circle or of four right angles. All the angles of any such figure are equal to twice as many right angles as the figure has sides minus four right angles, or if n be the number of sides, the sum of all the angles is $(2n - 4)$ right angles; hence each angle of a regular figure must bear the ratio $\frac{2n-4}{n}$ to a right angle, and as 4 right angles must be divisible by each of these angles, we have $\frac{4n}{2n-4}$, or $\frac{2n}{n-2} =$ an integer. But this number is equal to $2 + \frac{4}{n-2}$; hence $(n - 2)$ must be a divisor of 4, that is, either 1, 2, or 4, and n must therefore be either 3, 4, or 6; so that triangular, quadrilateral, and hexagonal regular polygons and prisms are the only ones which can collectively fill space without interstices. Now of these the one which has the greatest num-

ber of sides is the most economical. Hence the partition into regular hexagonal prisms is the most economical division of space in respect to two dimensions; but the third dimension is left undetermined.

If we suppose a hollow space, bounded by the surface of a regular hexagonal prism, to be open at one end, but closed at the other, so as to form a cell, it is clear that the base of this cell is most economized, if, like the sides, it be made also a boundary to another cell, or partially to several other cells. Hence, if two series of hexagonal cells, opening in opposite directions, have common bases, they present another feature of economy which is also exhibited in the form of the honeycomb. To this extent the bees' economy was known to the ancients.

There are two forms in which the bases of the cells might be fashioned so as to fit them as bases also of the opposite cells in the comb. They might be either plane hexagons perpendicular to the sides of the cells, or be composed of three rhombs inclined to the sides of the cells, and forming a solid angle in the central line or axis of the cell. If in the first form all the bases be in the same plane the relative positions of cells on opposite sides of the comb are immaterial to economy. But as each angle of the regular hexagonal prism is $\frac{4}{3}$ of a right angle, or $\frac{1}{3}$ of the circle, three regular hexagonal prisms have one common corner or line of contact, and this common corner must, in the second form of the bases, be in the same line with the axis of an opposite cell, the base of which is therefore composed of three thirds of the three contiguous bases.

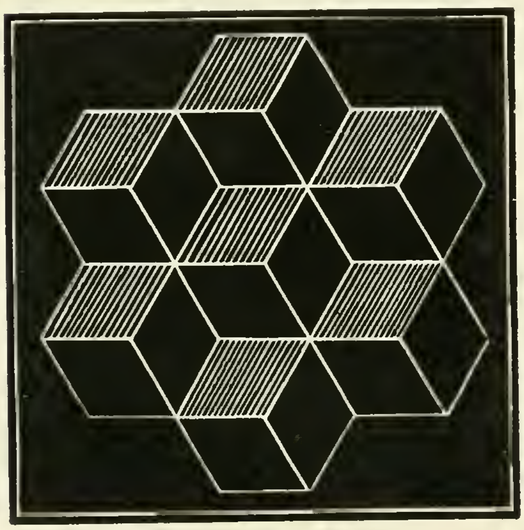


Fig. 1.

The dimensions of these rhombic segments of the bases vary with their inclinations to the sides and axes of the cells. One diagonal however is the same in position and length for all inclinations of the rhombs, while the other diagonal will vary with its inclination to the axis of the cell from the length of the shorter diagonal in the rhombs of a plane hexagonal base to any length whatever. This increase in the length of the variable diagonal increases the area of the rhombs, while the sides of the cells are more and more cut off and their area diminished as the rhombs become larger and more obliquely inclined.

In passing from the form of the plane hexagonal base, the sides at first diminish more rapidly than the rhombs increase, while the solid contents of the cell is the same for all the inclinations and corresponding forms of the rhombic segments of its base. It becomes a problem, therefore, of economy to determine that inclination and form of the rhombs which will give the least value to the sum of the areas in the sides and bases, or to the whole surface of the cell.

Instead of giving here the ordinary algebraic solution of this problem, which gives the same form to the rhombs as that determined by Maraldi's measurements of the honey-cell, we shall proceed upon the second line of argument to deduce the same form from considerations of symmetry, and we will then prove by geometrical reasoning that this form is a minimum. We may remark here, however, that Maraldi determined by measurement, not merely the angles of the rhombs, but also a symmetry in the structure, which, as some geometricians have supposed, might have enabled him by calculation to give these angles, 109° 28′ and 70° 32′, with so great precision. The symmetry which Maraldi observed was the equality of the angles that the planes of the rhombs make with each other to the angle 120° which the sides of the cell make with each other, and also with the rhombs. This symmetry depends

directly on the regularity of all the solid angles in the cell, which MARALDI also pointed out. Indeed, if we regard only the angles which the *planes* make with each other, there is but one angle, 120°, in the whole structure. This one symmetry affords a perfect rule to the insect architect.

If we wish further to determine the angles which the *lines* make with each other, it is not necessary to make any direct measurements, but only to calculate the angles at the vertex of a triangular pyramid of which the sides make with each other the angle 120°. These angles are incommensurate in the circle, but in the triangle they are as simple as the angles 120° and 60°; for while the cosines of the latter are $\mp \frac{1}{2}$ the former angles and their supplements (approximately 109° and 71°) have cosines exactly $\mp \frac{1}{3}$; and as the trigonometrical measurement of angles is the most easy and natural, these angles are as easily constructed as the others.

2. But why, it may be asked, should the bees' instinct prefer a structure, in which all the angles of the planes are 120°, to any other, unless this structure be also the most economical? That it is the most symmetrical structure of cells that can be imagined will be granted, in spite of its seeming complexity; but of what advantage is elegant symmetry to the bee, unless it also economizes labor and material? And what therefore could have fashioned the instinct of the bee except a supersensible principle of rational foresight, superior to mere sensible perception? In considering these questions let us look at the hive-cell from another point of view. Having built it up *a priori* from general principles, let us see in what relations it stands *a posteriori* to other cells.

We learn from naturalists that the cell is a species of nest or open cocoon, and that cocoons and nests are of two kinds, excavations and structures; but both kinds have these general characteristics of form. If closed, they are either spheres or cylindrical

figures with hemispherical ends; and if open, they are either hemispheres or cylindrical figures open at one end and terminated at the other by hemispheres. The latter form of cocoon, nest, or cell is then the natural type of the honey-cell. When finished the cells resemble a series of excavations in wax, though in fact they are structures. That a single excavation in sand or wood — a cylindrical pit terminated spherically — has even less surface for the same contents than the honey-cell, has never excited remark on account perhaps of other and more obvious utilities in such a simple symmetrical form.

If a series of such excavations be made as closely together as possible, there would result an arrangement like that of a pile of equal cylinders, each pit surrounded by six other pits, and though the surface of each cell would be enlarged in a greater proportion than the enclosed space by changing the figures of the excavations, still space would be gained, and, in case of a structure, material would be saved, by converting the cylindrical cells into regular hexagonal prisms, (as may be seen in Fig. 2); and this too by simple saving, or by the economy of afterthought. Again, if equal spherical excavations be made as closely together as possible there would result an arrangement like that of a pile of equal spheres (Fig. 2), each cavity surrounded by six others in the same layer,

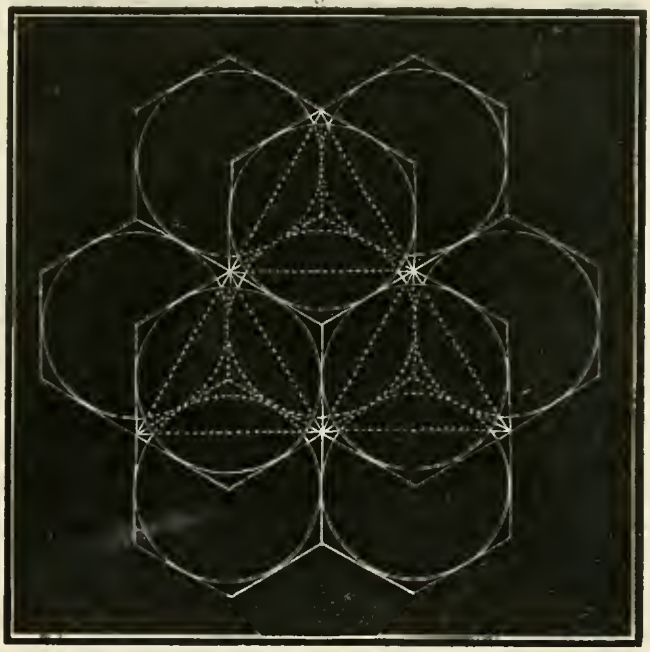


Fig. 2.

and by three others in each of the contiguous layers, — in all, by twelve spherical cavities. The walls that would be left by removing

the superfluous material of the interspaces or corners would bound regular hexagonal prisms, terminated at both ends by pyramidal bases (Figs. 4, 5), each of which would be composed of three rhombs of the same form as the rhombs of the honey-cell. If, therefore, we were to put into the cells of the honey-comb little spheres which would just fit them, and should press them to the bottoms of the cells on both sides of the comb, the spheres would touch the middle points of the rhombs and the middle lines of the sides, so that the sides and bases of the cells would be tangent planes to those points at which the spheres would touch each other but for the thickness of the intervening walls.

To prove this, let us consider two spheres in contact in two contiguous layers of a pile. It will be seen from Fig. 2 that the projections of the centres of two such spheres upon the ground plane are distant from each other by the length of a side of the circumscribed hexagon. If, therefore, in Fig. 3 we draw two vertical parallel lines, at the distance $O\,M$ apart, equal to the side of the hexagon circumscribed around the circle O, then

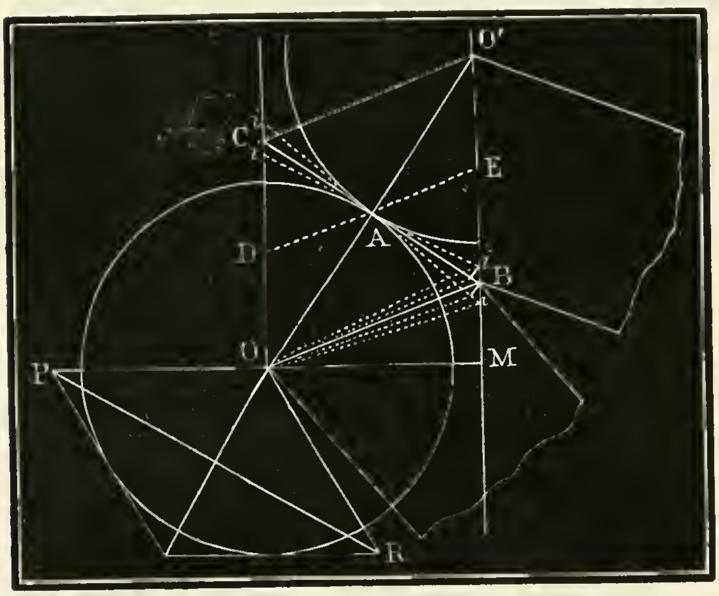


Fig. 3.

O and O' upon these two lines, at a distance apart of twice the radius of the circle, will represent the positions of the centres of the two spheres in contiguous layers. A, bisecting $O\,O'$, is the point of contact of the two spheres, and $B\,C$, perpendicular to $O\,O'$, is tangent to the two spheres, and is the shorter axis of the tangent rhomb. But $O\,O'$, equal to $P\,R$ or to twice the radius of the spheres,

is equal to the longer axis of the rhomb. Hence by drawing $C\,O'$ and $B\,O$ we have a figure, $C\,O\,B\,O'$, of the same form as the tangent rhomb. If this figure were turned around its shorter axis, $B\,C$, to a position perpendicular to the plane of the diagram, it would coincide with the tangent rhomb. Its dimensions can be readily computed.

If r denote the radius of the spheres or circles, then the side of the circumscribed hexagon is obviously $r\sqrt{\frac{4}{3}}$, and as $O\,O' = 2\,r$ we find $O'M = r\sqrt{\frac{8}{3}}$. As the ratio of the axes of the rhomb is equal to the ratio of $O'M$ to $O\,M$ or to $\sqrt{\frac{8}{3}} \div \sqrt{\frac{4}{3}} = \sqrt{2}$, we find $B\,C = \frac{2\,r}{\sqrt{2}} = r\sqrt{2}$. Hence the sides of the rhomb are equal to $r\sqrt{\frac{3}{2}}$. $B\,M$ is therefore equal to $r(\sqrt{\frac{8}{3}} - \sqrt{\frac{3}{2}}) = r\sqrt{\frac{1}{6}}$, and the ratio of $B\,M$ to $B\,O$ is $\sqrt{\frac{1}{6}} \div \sqrt{\frac{3}{2}} = \frac{1}{3}$, which is therefore the cosine of the angles of the rhomb. Hence these angles are the same as the angles in the rhombs of the honey-cell. This may be shown still simpler, without computation, by observing after Maraldi, that the solid angles of the honey-cell are all regular, that is, composed of equal plane angles. The triangular ones, which are formed by the intersections of the three rhombs and by the intersection of each rhomb with two sides of the cell, have the common plane angle 109° 28′ nearly, while the quadrilateral solid angles formed by the intersection of two rhombs and two sides have the common plane angle 70° 32′ nearly (see Figs. 4, 5). Now as all the spheres which can be drawn tangent to all the sides of a regular solid angle must touch them in lines that bisect them, it follows that a sphere touching all the sides and rhombs of the honey-cell must touch the rhombs in the intersections of these bisecting lines or in their middle points, and the sides in their middle lines. In fine, the form of the honey-cell is that which would be obtained by placing equal spheres in two layers, as in Fig. 2, and drawing tangent planes through their points of contact, and terminating these planes in

their mutual intersections. These planes may be regarded, without reference to the contact of spheres, as planes midway between equidistant points in two parallel series. They would therefore pass through the intersections of such equal spheres as have radii greater than half the distance between their centres. Fig. 3 also represents portions of two sides of the cell turned up into the plane of the rhomb around the corners $O\,B$ and $O'\,B$. It also shows that the angle made by the plane of the rhomb, or by its shorter axis with the axis of the cell, is half the larger angle of the rhomb.

We have thus determined the form of the honey-cell from principles of symmetry alone. That this form is the most economical is easily shown by supposing it to vary, and by determining the changes of area in the sides and bases.

Let us suppose, as before, the rhomb $C\,O\,B\,O'$ to be turned upon the axis $B\,C$ by a right angle, so that the corners O and O' may fall in direction upon A, and let the sides that are attached be turned downward into their positions as vertical planes tangent to the sphere around O. In turning the rhomb now around its longer axis, the corners C and B must remain in the vertical lines $C\,O$ and $B\,O'$, so that the shorter axis is diminished if B is raised, and increased if B is depressed. Let i be the amount by which B is raised or depressed; then the projection of i upon the shorter axis $B\,C$ of the rhomb is half of the amount by which this axis is made shorter or longer. The projection of i upon $B\,C$ reduces it in the proportion of the semi-minor axis to the side of the rhomb or by the ratio $r\sqrt{\frac{1}{2}} \div r\sqrt{\frac{3}{2}}$, hence the projection is $i \times \sqrt{\frac{1}{3}}$. Half of this multiplied by the semi-major axis r is the change of area on each edge of the rhomb; hence the whole rhomb is decreased or increased by $2\,r\,.\,i\,.\sqrt{\frac{1}{3}}$. But $\frac{1}{2}\,i$ multiplied by the breadth of the side of the cell is the corresponding change in each of the sides attached to the rhomb, and hence both sides together are increased or decreased by

$i \times r\sqrt{\frac{4}{3}} = 2\,r\,.\,i\,.\sqrt{\frac{1}{3}}.$ The sides gain as much therefore as the rhombs lose, or lose as much as the rhombs gain, by an infinitesimal change in their positions and form; and the whole surface is therefore either a *maximum* or a *minimum*, while the contents of the cell is unchanged by such a change of form.

Again, the infinitesimal triangles on either side of the line $B\,C$, resting on the bases i, are the amounts which the sides of the cell are made to gain or lose, while the smaller infinitesimal triangles on either side of $B\,C$, and resting on the projections of i, are the amounts which the edges of the rhombs lose or gain. These smaller triangles are in vanishing equal to half the larger ones, since the infinitely smaller right triangles resting upon i are similar to $O'A\,B$, the hypothenuse of which, $O'B$, homologous to i, is bisected by the line $D\,E$, and to this line the sides of the smaller infinitesimal triangles become parallel. If i be finite, the decrease of the minor axis of the rhomb will be less than twice the projection of i upon $B\,C$, and its increase will be more than twice this quantity; hence the rhomb will lose less than the sides gain, or gain more than the sides lose, by a finite change in the positions and form of the rhombs. The whole surface is therefore a *minimum*. Thus the symmetrical form of the honey-cell is seen to be also a *minimum* form.

This economy however has no reference to the depth of the cell, so that geometricians have conceived of a further limitation of the cell's form, which is not observed by the bee. Having determined the most economical arrangement of the sides and bases, we may further limit the length and breadth of the sides to that ratio, which gives what has been called and determined as the *minimum minimorum* form of the cell. If A denote the whole area of the cell's surface, C the whole solid contents, and l the length of the longer edges of the cell, or of those which terminate in the longer axes of the rhombs, (Figs. 4, 5,) and if, as before, we denote by r the radius of

the inscribed sphere or cylinder, then from the dimensions already computed we find

$$A = 2\,r^2 \sqrt{2} + 4\,lr \sqrt{3}, \qquad C = 2\,r^2 l \sqrt{3}.$$

If one of these be supposed constant and the other a minimum, their derivatives must both be equal to zero. Hence by differentiation and reduction we have the equations

$$r\sqrt{2} + r\,D_r l \sqrt{3} + l\sqrt{3} = 0, \qquad 2\,l + r\,D_r l = 0,$$

from which, eliminating $D_r l$, we obtain $l = r\sqrt{\tfrac{2}{3}}$. Hence, l in the *minimum minimorum* cell is half the distance $O'M$ between the two planes on which the centres of the contiguous inscribed spheres are arranged. This form of the honey-cell is therefore what the bee would fashion if, instead of its deep nests, it should construct hemispherical nests like the bird, and then convert them into polyhedral figures with sides tangent to the contiguous hemispheres on both sides of the comb. Two such cells would exactly enclose a sphere with tangent planes, as in Figs. 4, 5. Of all figures, therefore, which can divide space into equal and similar parts, without interstices, the polyhedrons of Figs. 4, 5 have the least surface for a given contents.

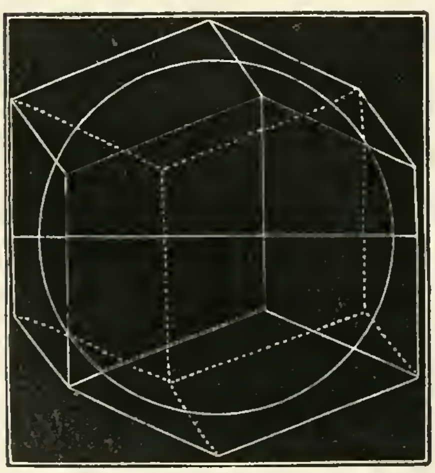


Fig. 4.

These figures are such as would be formed by drawing tangent planes through the twelve points of contact on a sphere surrounded by twelve equal spheres. The two forms arise from the two ways in which the three spheres, in each of the outside layers, may be arranged. It may be seen in Fig. 2 that the three superposed spheres occupy alternate interstices; so that there are two ways in which they may be placed in contact with the central one. If the three spheres, which are in contact with the central one from below, be opposite

the three superposed ones, we have the polyhedron of Fig. 5; if alternately disposed, we have that of Fig. 4, the sides of which are all equal and similar rhombs, like the rhombs of the honey-cell. Fig. 5 may be produced by turning the lower half of Fig. 4 one sixth of its circumference around its vertical axis.

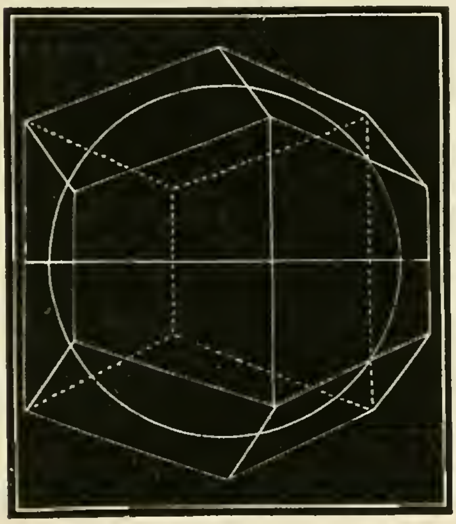


Fig. 5.

Geometricians have also criticised the economy of the bee by computing the difference between the surface of cells with plane bases and the surface of honey-cells having the same contents; and by showing that this difference is only about $\frac{1}{50}$ part of the whole surface of a completed cell; so that the bee is enabled by the pyramidal form of the base to make 51 completed cells, instead of 50, out of the same material, — a gain hardly worth so great precision in construction.

3. But this criticism would be just only on the suppositions, 1st, that a plane base could be more readily made than a pyramidal one, and 2dly, that the instinct of the bee is determined to a structure the most economical *a priori* and on the whole. These suppositions are not however necessary, nor are they supported by facts.

It is true, indeed, as we have seen, that the form of the honey-cell requires less material than any similar one for the construction of a double series of cells like the honeycomb. But we have also seen that simple nests are bounded by still less surface than the honey-cell, in proportion to their contents, and that the deviation of the single honey-cell from the cylindrical nest with a hemispherical base does in fact increase the surface in a greater proportion than it increases the enclosed space, — with a gain however of material when the cells are contiguous. As their bases are boundaries be-

tween cells on opposite sides of the comb, the pyramidal form would be possible as a modification of the hemisphere only in case of a close symmetrical arrangement of the opposite cells. But for such an arrangement, — of two series of spherical bases in closest contact, — a modification to the pyramidal form of the honey-cell is not merely the most economical, but also the simplest, and one perfectly analogous to the hexagonal modification of the sides.

That the cell ought to be regarded as such a modification of the simple nest by *simple* economy will be evident when we consider the mode of its construction. If the cells were excavated nests made in a costly material, they would by the closest arrangement and the removal of the interstitial material, receive through the agency of simple economy the hexagonal form of the honey-cell; but the bases of opposite cells could only by accident meet as they do in the honeycomb, and hence they would either retain, without reference to each other, the hemispherical form, or be flattened to plane hexagonal bases. But the hive-bee does in fact build up its nests from their bases which are rudely fashioned at first on the edges of the comb in the form of little cavities at equal distances. On one side of the comb these cavities form by their depressions the interstitial elevations between the cavities of the other side; and hence the symmetrical arrangement of the two series of cells, and hence also the pyramidal form of the bases and their economy.

It has been observed that mechanical pressure alone is capable of changing the elongated cells of cellular tissues — as in the fruits and piths of some plants — into the polyhedral form of the honey-cell. This however is not the way in which the honey-cell is formed, for the bees fashion the corners of their cells as they build, by continual trimming and saving.

Again, the bee builds with great precision; indeed, the exactness of the structure is more surprising than its economy. But precision

is not required by economy at those limits of form which determine *maxima* and *minima* values; for these values are determined by the condition that they shall vary least by slight changes in the forms on which they depend. Symmetry, on the contrary, requires absolute precision and affords the means of effecting it. Thus, when the bee's point of view is just over each one of the middle points of the rhombs it ought to be at the same distance from all the nine planes of the cell, and just opposite similar points of view in the nine contiguous cells. This symmetry affords a working guide to the bee as perfect as the regularity of the solid angles or the equality of all the angles made by the planes with each other.

It is true that these symmetries also determine the economy by which the honey-cell is characterized; so also does the still simpler symmetry of the cylindrical and hemispherical nest determine a still greater economy of work and surface in the architecture of innumerable other nest-building animals. But have psychologists, therefore, thought it necessary to regard the instincts of these animals as determined by supersensible properties of form, instead of that facility of construction and simple saving which are apparent to the senses?

Many deviations not only from the strictest economy, but also from symmetry, are observed in the honeycomb, and are required by the various sizes and uses of the cells.

In the normal form of the comb the triangular corners are formed by the meeting of four cells or by the intersection of six planes, three belonging to the bases and three to the sides; while the quadrilateral corners are formed by the meeting of six cells, or the intersection of twelve planes, six belonging to the bases and six to the sides. Now the bees very frequently in finishing their cells do not make these twelve planes meet in a common point, but connect them by a thirteenth very small plane, forming a tenth partition or

a fourth basal segment to two such cells on opposite sides of the comb as would otherwise only touch each other by a common corner. This supplementary basal segment is however perpendicular to the line which joins the symmetrical centres of the two cells, and is therefore easily included in the symmetries of the cell by the following considerations.

We have seen that if the lines joining equidistant points in two parallel planes be bisected by planes perpendicular to them, these planes terminating in their mutual intersections will form the corners of the honey-comb. Such lines of juncture form the edges of the regular tetrahedrons and octohedrons into which the space between the two planes may be divided; each tetrahedron surrounded by three octohedrons, and each octohedron by six tetrahedrons. The six planes that bisect the six edges of each tetrahedron meet in the centre of the figure and form four contiguous triangular corners. The twelve planes that bisect the twelve edges of each octohedron form in the centre of the figure six contiguous quadrilateral corners; and if in addition to these planes three be drawn midway between each pair of diagonal corners in the octohedron, they will also pass through the centre of the figure, but would be of no service as partition walls between these corners in a perfectly regular figure. If, however, the points in the parallel planes be not exactly equidistant, and the included figure vary from a regular octohedron, the twelve planes cannot at the same time bisect the edges of the figure and meet in the centre; and the bee accordingly, forced by this necessity, instead of making these planes meet in a common point, chooses to join them by another small one midway between two diagonal corners of the figure. Thus the bees separate the central points of their nests by planes midway between the centres of contiguous nests on the same side and on opposite sides of the comb, and also in irregular structures, by smaller planes

between the centres of such nests on opposite sides of the comb, as would otherwise only touch each other in a common quadrilateral corner. This way of remedying irregularities of structure shows conclusively that the bee's instinct is chiefly governed by *symmetries of partition* instead of those that belong to any particular figure.

Though it is to be presumed from what we have seen, that special facilities and economies of labor and material govern all abnormal forms, yet habit, undoubtedly, must greatly modify them from special adaptations to general conformities of structure.

The symmetries which we have studied are obviously not the same to the bee and to the geometrician, for what the latter comprehends abstractly as symmetry of form, is only perceptible to the former concretely, as facility of construction. Like the two kinds of economy, the one is rational, the other sensible.

An unreflective and unforeseeing economy, which, without reference to an end, simply saves, through sensuous preference, what the conditions of life render useful and costly to the race, characterizes the whole animal kingdom. For even those animals which do not store food or build structures for themselves or their offspring, still select through sensuous preference the food that is nutritious and appropriate to them, without foreseeing its use. An application of such a simple saving to the typical structure of the nest results in the honey-cell. The bee's instinct ought not therefore to be regarded as an exception to animal instincts in general; much less ought animal instincts in general to be so interpreted as to include a misconception of the bee's instinct.

A natural choice from simple feelings and from heritable sensuous associations — empirical in form, whatever their origin — is all that is required to account for the most perfect work of instinct.

A SECOND BOOK IN GEOMETRY.

[Continued from Page 143.]

CHAPTER VIII.

THE MAXIMUM AREA.

114. I will prove only one more proposition; but I will select a difficult one, in order that it may require a number of preliminary proofs. I will select the proposition given in the "First Lessons in Geometry," chap. xxiii. § 14: — "Of all isoperimetrical figures the circle is the very largest."

115. When we attempt to analyze this, we shall see that it implies that any regular polygon is less than a circle isoperimetrical with it, and that any other polygon is less than a regular one, isoperimetrical with it.

116. Let us begin, however, by defining a few of the words we shall need to use.

117. A polygon is a plane figure bounded by straight lines.

118. The perimeter of a polygon is the sum of the length of its sides.

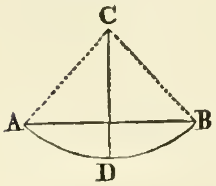
119. Isoperimetrical polygons are those of equal perimeter.

120. Among quantities of the same kind, the largest is called a maximum.

121. A circle is a plane figure, bounded by one line that curves equally in every part. This line is called the circumference of the circle, and frequently the line itself is called the circle. Portions of the circumference are called arcs.

122. *Theorem.* There is a point within the circle equally distant from every point of the circumference. This point is called the centre of the circle. — *Proof.* Let A D and B D be equal adjacent arcs in a circumference. Through the points A, B, and D draw lines at right angles to the curve at those points. Now, since the circle curves uniformly at every point, the point of intersection with B C must be the same point in both lines, A C and D C, and the points A, B, and D are equally distant from C. But A and B may be taken anywhere in the circle, only provided they are equally distant from D; and hence every point in the circle is equally distant from C, the centre of the circle.

Fig. 60.



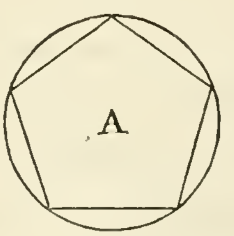
123. A straight line joining the centre to the circumference is called a radius. The straight line formed of two opposite radii is called a diameter.

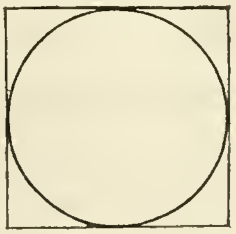
124. All radii are of course equal to each other.

125. A straight line joining the two ends of an arc is called a chord.

126. A straight line which, however much prolonged, touches the circle in one point only, is called a tangent.

127. It is manifest that the tangent coincides in direction with the arc at the point of contact.





128. A polygon formed wholly of chords in a circle is said to be inscribed in that circle.

129. A polygon formed wholly by tangents to a circle is said to be circumscribed about the circle.

130. The circle is said to be inscribed in the circumscribed polygon, and to be circumscribed about the inscribed polygon.

131. If a polygon, about which a circle can be circumscribed, or in which a circle can be inscribed, has its sides equal, one to the other, the polygon is called a regular polygon, and the centre of these circles is called also the centre of the polygon.

132. Let us now attempt to analyze the proposition that the circle is the maximum among isoperimetrical polygons. This is equivalent to saying that if an isoperimetrical circle and regular polygon are laid one over the other, the polygon will be the smaller. But we see that by laying them one on the other, a circle inscribed in the polygon would be smaller than the isoperimetrical circle. The question of course suggests itself, whether the area of a regular polygon is not proportional to the radius of the inscribed circle. Now it is plain that it is. For by dividing the polygon into triangles, by lines from its centre to its vertices, we find the area of the polygon will be the sum of the areas of the triangles, and these areas will be measured by half the product of the perimeter multiplied by the radius of the inscribed circle. The area of the isoperimetrical circle will be measured by half the product of the perimeter multiplied by the radius. But as the perimeters of the polygon and the isoperimetrical circle are the same, and the radius of the inscribed circle is smaller than that of the isoperimetrical circle, it is evident that the area of the polygon is smaller than that of the isoperimetrical circle.

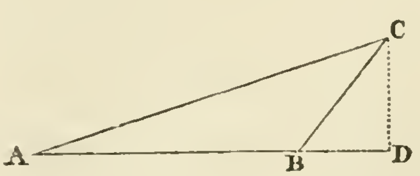
It will now remain to show that the area of a regular polygon is greater than that of an isoperimetrical irregular polygon. It is evident that this can be done, since a polygon of given sides is manifestly largest when most nearly circular, and a polygon of given perimeter inscribed in a circle is manifestly largest when the sides are equal. We can surely have no difficulty in proving these two points, and then our proof will be complete.

133. Let us return, then, to the synthetic mode, and establish these propositions: — First, that the maximum of polygons formed of given sides may be inscribed in a circle; secondly, that the maximum of isoperimetrical polygons having a given number of sides has its sides equal; and thirdly, that such a regular polygon is of smaller area than a circle isoperimetrical with it.

134. *Theorem.* The area of a triangle is found by multiplying the base by half the altitude. This theorem has been already proved (Art. 111).

135. We shall need the Pythagorean proposition, which implies all the propositions into which we have already analyzed it (Arts. 64 – 113.)

136. *Theorem.* Of two unequal lines, from a point to a third straight line, the shorter is more nearly perpendicular to the third line. — *Proof.* Let C be the given point, and A D the third straight line. Let C A and C B be two lines, of which C B is the shorter. Draw C D perpendicular to A D. We wish to prove that B D is shorter than A D. But this is manifest from the Pythagorean proposition, since the square on A D is the difference of the squares on A C and C D, and the square on B D is the difference of the squares on B C (which is smaller than A C) and the same C D.

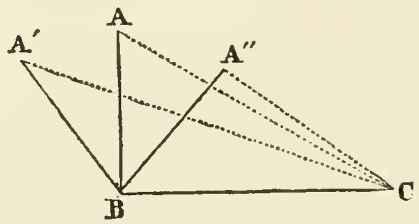


137. *Corollary.* A perpendicular is the shortest line from a point to a given straight line.

138. *Theorem.* The maximum of triangles having two sides given is formed when these two sides are at right angles. — *Proof.* Let A′ B, A B, and A″ B be equal to each other. The area of A′ B C, A B C, or A″ B C, being found by multiplying B C into the perpendicular height

of A, A′, or A″, above B C, will be in proportion to that height. (Fig. 17.) Let then A B be perpendicular to B C, and the height of A above the base will equal B A. But the height of A″ above the base must, by 137, be less than B A″ which is equal to B A.

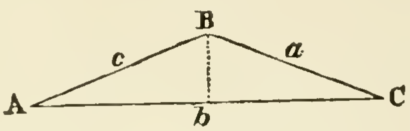
Fig. 17.



139. An angle is said to be measured by an arc of a circle such as would be intercepted by radii making that angle with each other. And, since the circumference curves equally in all parts, and the radii are at right angles to it, it is evident that this measure is just, and that the angle will bear the same ratio to four right angles that the arc bears to a whole circumference, whatever be the size of the circle.

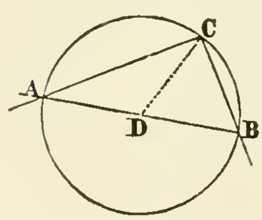
140. *Theorem.* If two sides in a triangle are equal, the angles opposite those sides are equal. — *Proof.* Let A B and B C be equal sides in a triangle. Imagine A C divided in the centre, at the point *b*. The triangles A B *b* and C B *b* will now be composed of equal sides, and we have already proved (Arts. 91 – 95) that they must have equal angles, — that is, the angle at A is equal to that at C.

Fig. 16.



141 *Theorem.* If one side of a triangle is prolonged, the external angle is equal to the sum of the opposite internal angles. This has been proved in Art. 57.

142 Two chords starting from one point in a circumference intercept double the arc that would be intercepted by radii making the same angle; or, the angle of the chords is measured by half the arc included between them. — *Proof.* If one chord, as A B, passes through the centre D of the circle, it is plain that by drawing D C the angle C D B will be equal to the sum of the angles C A D and D C A. But since D A and D C are equal, these angles are equal, and C D B is equal to twice C A D.

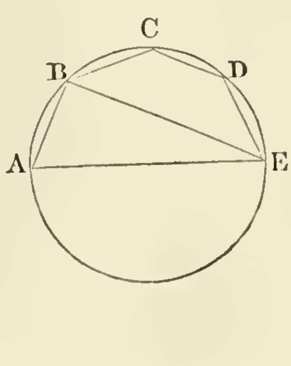


If neither chord passes through the centre of the circle, we can draw a third chord, starting from A, passing through the centre of the circle, and apply this reasoning to the two angles formed with this third chord by the other two. The angle of the other two chords will simply be the sum or the difference of these two angles.

143. *Corollary.* If the vertex of a right angle be placed in the circumference, the sides will intercept a semicircle.

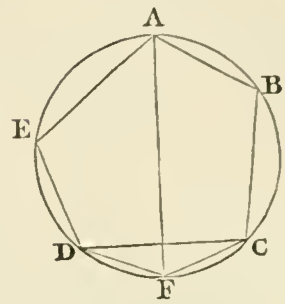
144. *Corollary.* If a circle be circumscribed about a triangle, and one side of the triangle passes through the centre of the circle, the opposite angle is a right angle.

145 *Theorem.* The maximum of polygons, having all the sides given but one, may have a circle circumscribed about it, having the unknown side for a diameter — *Proof.* Let A B C D E be the maximum polygon, formed of given sides, A B, B C, &c., and the unknown side, A E. Join B E by a straight line. Now, since the polygon is a maximum, we cannot, leaving B E unaltered, enlarge the triangle A B E, because that would enlarge the polygon. The angle A B E is therefore a right angle, by Art. 138, and a circumference, having A E for its diameter, would pass through the point B. In like manner it can be shown that a circumference having the same diameter would pass through each of the other points.



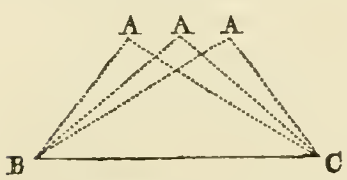
146. *Theorem.* The maximum of polygons formed with given sides can be inscribed in a circle. — *Proof.* Let A B C D E be a polygon formed of given sides, with a circle circumscribed about it. Draw the diameter A F, and join F C and F D. The polygons A B C F and A D E F are now maximum polygons, and therefore A B C D E must also be a maximum, since its enlargement would enlarge the sum of the other two.

Fig 9.

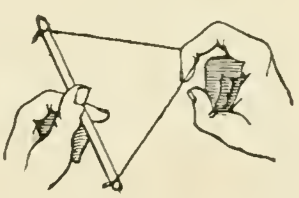


We have thus proved the converse of the proposition, and the proposition is true, unless there is more than one maximum form of the polygon.

The converse is more easily proved than the proposition, and I therefore proved it, on the assumption that there is but one maximum form. That is, I have proved that a polygon of given sides, when inscribed in a circle, is a maximum; but that does not strictly *prove* that the maximum can always be inscribed in a circle; except on the assumption, which is, however, a safe one, that a polygon formed of given sides, arranged in a given order of succession, can have but one maximum form.



147. *Theorem.* Of isoperimetrical triangles with one side given, the maximum has the two undetermined sides equal. — *Proof.* In order to prove this we have only to show that the point A is at its greatest distance from the base, B C, when opposite the middle of it. This might seem scarcely to need proof. For when we use a string and stick to illustrate the problem, we can see that by sliding the finger from the middle of the string, it can be brought down into a line with the stick; and the greatest height from the stick is near the middle of the string. Further consideration shows it must be exactly at the centre of the string, because the finger and string have precisely the same relation to one end of the stick as to the other; and a motion towards either end must affect the height of the finger in a similar manner.

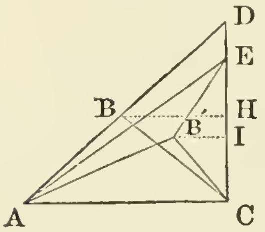


This reasoning is doubtless satisfactory to every fair mind. Yet it is not a good mathematical demonstration, and I have given it to you for the purpose of illustrating the peculiar nature of mathematical reasoning. The reasoning just given leaves no real doubt on the mind, but it is rather because we *see* with the eye that the finger is highest in the middle, than because we see with the mind that it must be. There is another step still lacking, to prove to us that the highest points are not on each side of the exact middle, as that would satisfy the conditions of symmetry and of declination towards each end. Let us then seek a proof which shall not force us to consider the whole motion of the finger, but which shall simply compare two forms of the triangle, one with the finger in the middle of the string, and one with the finger on one side.

148. *Theorem.* If a straight line be drawn from the vertex of two equal sides in a triangle, at right angles to the third side, it divides the third side into equal parts. — *Proof.* Let c and a be equal sides in a triangle A B C. Since (Fig. 16, Art. 140) the angles at A and C are equal, the angles b B C and b B A are also equal. If, therefore, the triangle B b c be folded over on the line B b, the line a will take the same direction as the line c, and, being of the same length, will coincide with it. Hence, b C will also coincide with b A, and the two lines must be of equal length.

149. New proof of Art. 147. Let A B C and A B′C be isoperimetrical, and let A B and

B C be equal. Continue A B to D, making B D = B A = B C, and join D C. Then, by Art. 144, the angle D C A is a right angle. Draw B′ E making it equal to B′ C. Join A E. A E will be less than the sum of A B′ and B′ E, that is less than A B′ and B′ C, that is less than A B and B C, that is less than A D. But if A E is less than A D, then C E must be less than C D, because the square on C E is equivalent to the difference of the squares on A E and A C, while the square on C D is equivalent to the difference of the squares on A D and A C. Draw B H and B′ I at right angles to C D; we have C I which is half C E less than C H which is half C D. But C I and C H are the altitudes of the triangles A B C and A B′ C above their common base A C. The triangle with the undetermined sides equal has the greatest altitude and must be the largest triangle.



150. *Theorem.* The maximum of isoperimetrical polygons of a given *number* of sides is equilateral, that is, has equal sides. — *Proof.* Let A B C E D (Fig. 9) be the maximum of isoperimetrical polygons of a given number of sides. Then A B must equal B C. For if it did not, then after joining A and C we could enlarge the triangle A B C by equalizing A B and A C, and thus enlarge the polygon without altering the number of sides or the perimeter, and the present form would not be the maximum.

In like manner we may prove that B C = C E, &c.

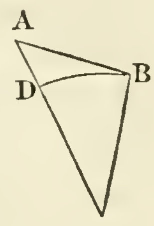
151. *Corollary.* The maximum of isoperimetrical polygons of a given number of sides is regular by Arts. 150 and 146.

152. *Axiom.* A circle may be considered as a regular polygon having an unlimited number of sides. And this regular polygon may be considered as either inscribed in or circumscribed about the real curve.

153. *Theorem.* The area of a regular polygon is measured by half the product of the perimeter into the radius of the inscribed circle. — *Proof.* For if lines be drawn from each vertex of the polygon to the centre, the polygon will be divided into triangles having a common altitude equal to the radius of the inscribed circle, the sum of the bases of these triangles being equal to the perimeter of the polygon.

154. *Corollary.* The area of a circle is measured by half the product of the radius into the circumference.

155. *Theorem.* The perimeter of a circumscribed polygon is greater than the circumference of the circle. — *Proof.* Let A B be half a side of a circumscribed polygon, and D B the portion of arc intercepted by lines drawn from A and B to the centre of the circle. Divide D B into arcs so small that each may be considered as a short straight line. Through the points of division draw lines extending from the line A B to the point C. At the end B, the little arcs are equal to the corresponding pieces of the line A B; but as you approach A the divisions of the line grow longer than the corresponding divisions of the arc, for two reasons; first, the little arcs are at right angles to the radii, while the portions of the line are not (Art. 136); secondly, the little arcs are nearer to the point C, towards which the radii converge. The whole of A B must therefore be longer than the whole of D B. But it is manifest that the circumference consists of as many times D B as the perimeter does of the line A B.



156. *Corollary.* The circle inscribed in a regular polygon is smaller than a circle isoperimetrical with the polygon and has a shorter radius.

157. *Corollary.* The circle is the maximum among isoperimetrical regular polygons.

158. *Corollary.* The circle is the maximum among isoperimetrical figures; a proposition

towards which we have been directing our course through 48 articles, some of which are themselves complex propositions referring to the preceding chapters. No other science requires anything like such long trains of connected reasoning, as those used in the mathematics. An argument in other matters usually consists of only a few steps; — what are called long arguments being really a collection of shorter independent proofs of the same thing. In the mathematics we are frequently required to take, as in the present instance, hundreds of consecutive steps to attain a single position.

159. *Scholium.* A slight modification of the reasoning in Arts. 155 – 157 would show that of isoperimetrical polygons that is greatest which has the greatest number of sides.

160. He that really wishes to learn geometry must learn to work alone. I advise the learner now to take up "First Lessons in Geometry," and, beginning with the fourth chapter, go through to the twenty-sixth, trying how many of the facts he can prove. I think he can, if he sets himself to work with a good will, prove the greater part. Perhaps he will be obliged to ask some help of his teacher, but I think not much. He will, however, do well to show his demonstrations to his teacher for his criticism.

When he comes to Chap. XXVI. of the "First Lessons" he will be obliged to lay down the book again, as the propositions in the remainder of the book cannot be proved without the aid of higher branches of mathematics, — Algebra, Trigonometry, and the Calculus.

161. In proving the Pythagorean proposition, and the proposition that the circle is the maximum among isoperimetrical plane figures, I have tried to give good examples of the mathematical mode of proof; — the analysis, in which the mind turns the proposition over in every form, trying all sorts of experiments upon it intellectually, to discover its vulnerable side; — and the synthesis, by which we then enter step by step into the very secret of the mystery.

Analysis consists in taking the proposition itself as the starting point, and going, step by step, to self-evident truths, or at least to truths already proved. Synthesis consists in starting with self-evident truths, or truths already proved, and going step by step to the truth which you would prove. But synthesis generally requires a previous rough analysis by which you select the proper point of departure for your synthetical reasoning.

A species of analysis called *reductio ad absurdum* is often used, in cases where true analysis, or true synthesis, is difficult. In this form of proof you assume that your proposition is not true, and by analysis show that this would lead you, step by step, to the denial of self-evident truth. This shows the proposition to be true, by simply showing that it cannot be false. Article 138 gives an instance of this proof.

I think you will, after mastering this book thoroughly, be able to read any of the books on geometry which you will be at all likely to meet with.

END OF PART I.

Mathematical Monthly Notices.

The Ladies and Gentlemen's Diary, or United States Almanac, and Repository of Science and Amusement. Intended for an Annual Magazine, including a variety of matter, chiefly original, on subjects of general utility, in the Arts, Sciences, Agriculture, Manufactures, &c., &c. Edited by M. Nash. New York: J. Seymour, Printer. 1819.

The work is divided into two parts. The former entitled the Ladies and Gentlemen's Diary,

or United States Almanac, is designed, as stated in the address to the public, "to provide for the friends of science a convenient guide and companion, which will assist those already skilled in the science of Astronomy, and instruct those who have had less opportunities for information. To offer superior advantages to ingenious persons of every description, particularly to such as are employed in coasting vessels, for an improved knowledge of the tides, and to whom a greater acquaintance with the stars must not only be agreeable, but of eminent utility — to be in many respects a national work — an Almanack for the United States. The latter, entitled the Repository of Science and Amusement, "will include original essays, relating to the arts and sciences; and every year mathematical questions to be answered in the next. Then will follow literary and philosophic queries, poetical enigmas, rebusses, and charades; likewise to be answered in the succeeding year."

The mathematical department was mainly enriched by the contributions of Prof. ADRAIN, JOHN MACAULEY, of Liverpool, England, the Hon. WALTER FOLGER, JR., Member of Congress from Nantucket, GEORGE GARTH, Philadelphia, JOHN CAPP, Harrisburgh, Penn., and JAMES PHILLIPS, Haerlem, N. Y., now Prof. of Mathematics in the University of North Carolina. GEORGE GARTH, of Philadelphia, was by far the most successful competitor in the solution of the Philosophic Queries.

Three numbers only of this highly interesting and instructive work were published. The first number appeared in 1820. — WILLIAM TIMPSON, White Plains, N. Y.

An Elementary Treatise on the Lunar Theory, with a brief sketch of the history of the problem up to the time of NEWTON. By HUGH GODFRAY, M. A., of St. John's College, Cambridge. Second edition, revised. MacMillan and Co., Cambridge, and 23 Henrietta Street, Covent Garden, London. 1859. pp. 119.

This work is intended as introductory to the more elaborate treatises on the Lunar Theory, and we heartily recommend it to those students who wish to understand the general character of this important problem. The first chapter is devoted to the "Principle of Superposition of Small Motions, and "Attractions of Spherical Bodies." The problems of two and three bodies are considered in the second chapter; and in the third, fourth, and fifth chapters we find the rigorous differential equations of the Moon's motions, their integration to quantities of the second order, and the methods which may be employed for computing the numerical values of the coefficients. The sixth chapter gives the physical interpretation of the results, the seventh, a series of propositions connected with the processes in the Lunar Theory, and the eighth, the history of the lunar problem before NEWTON. This little volume is an admirable introduction to the study of this great problem.

A Treatise on Conic Sections; containing an account of some of the most important modern algebraic and geometric methods. By REV. GEORGE SALMON, A.M., Fellow and Tutor, Trinity College, Dublin. Third edition, revised and enlarged. London: Longman, Brown, Green, and Longmans. 1855.

An Elementary Treatise on Conic Sections and Algebraic Geometry; with a numerous collection of easy examples, progressively arranged. Especially designed for the use of schools and beginners. By G. HALE PUCKLE, M. A., St. John's College, Cambridge, Principal of Windemere College. Second edition, enlarged and improved. MacMillan and Co., Cambridge, and 23 Henrietta Street, Covent Garden, London. 1858.

We have brought these two works together for the reason that the latter is written upon the same plan as the former, and is essentially introductory to it; and therefore what we have to say applies equally to both. It is hardly to be supposed that two such admirable works as these before us should be entirely unknown in this country after so many years' publication;

but we have no reason to suppose that they are by any means so generally known as to render a brief notice unnecessary.

We have already called the attention of our readers to two of Mr. Salmon's books, "A Treatise on Higher Plane Curves," and "Lessons on Higher Algebra," and we believe we shall be doing mathematical students a good service by urging them to give the present volumes their special attention. For those who have already made some progress in the study of Analytic Geometry Mr. Salmon's work will be sufficient. One most valuable feature in the plan of these works is the collections of problems which immediately follow the discussion of each proposition, to test the student's understanding of the subject as fast as it is gone over; the former contains over 400, and the latter nearly as many. In Mr. Salmon's book many of the problems are solved, and to a much larger number are appended such remarks and suggestions as will enable the student to master them. Mr. Puckle gives the answers to all the questions at the end of the volume. But excellent as these volumes are in this respect, and also in the arrangement of the propositions, and the elegance and clearness of their demonstrations, we only call attention to them with regard to a want which the text books on the subject in use in this country do not supply. These volumes, the former more fully, give a systematic and intelligible account of "the algebraic and geometrical methods which have been introduced into use of late years;" but upon which our own text books are silent. Perhaps we shall make our idea more clear by specifying some of the methods. In the first place there is an "abridged notation" which very much facilitates otherwise long and tedious algebraical computations. Thus, if $\alpha = 0$, and $\beta = 0$, are the equations of two straight lines, then $\alpha - \beta = 0$ is that of a line passing through the intersection of $\alpha = 0$, $\beta = 0$, bisecting the angle they make with each other. If, then, $\alpha = 0$, $\beta = 0$, $\gamma = 0$, denote the three sides of a triangle, $\alpha - \beta = 0$, $\beta - \gamma = 0$, $\gamma - \alpha = 0$, denote the equations of the bisectors of the angles of this triangle, and since their sum vanishes identically, we have the proof that the bisectors of the angles of a triangle meet in a point. Again, if $S = 0$ and $S' = 0$ are the equations of two conics, then $S + k\,S' = 0$ is that of the conic passing through the four real or imaginary points in which $S = 0$ and $S' = 0$ intersect. If S' breaks up into two right lines $\alpha = 0, \beta = 0$, then $S + k\,\alpha\,\beta = 0$ is the conic passing through the four points in which α and β intersect S. If both S and S' break into two right lines each, then $\alpha\,\gamma + k\,\beta\,\delta = 0$ denotes the conic circumscribing the quadrilateral $(\alpha\,\beta\,\gamma\,\delta)$. But we have no room for further illustration of the facility with which such a notation can be applied, to render propositions, quite difficult by the usual methods, almost self-evident.

We must refer the student to the volumes themselves, especially the former, for the methods of "Reciprocal Polars," "Harmonic and Anharmonic properties of Conics," "Involution," "Projections," &c. Indeed, some of the chapters of these books contain a new language for those only acquainted with the text books on Analytic Geometry of fifteen or twenty years ago. Those teachers and students who are desirous of becoming acquainted with the modern methods will never regret having had their particular attention called to these volumes.

Théorie Général de l'Élimination. Par Le Chevalier François Faà de Bruno, Docteur ès Sciences de la Faculté de Paris, Capitaine Honoraire de l'État Major Sadre, Professeur Libre a l'Université de Turin. Paris: Libraire Centrale des Sciences, Leiber et Faraguet, Rue de Seine-Saint-Germain. No. 13. 1859.

There have always been two different classes of workers in every department of science, the inventors, and those who come after and systematize and popularize all that is known in any particular department. And while the former have a more direct agency in the progress of science, it may well be doubted whether the latter by the broad culture which is the inevi-

table result of their labors, are not, after all and on the whole, its greatest promoters, by preparing the way for future discoveries.

In this volume the labors of BEZOUT, JACOBI, BETTI, LIOUVILLE, CAYLEY, SYLVESTER, BORCHARDT, &c., which relate to elimination, but scattered through the various Journals and Proceedings of Academies, are co-ordinated under a single point of view. The aid thus rendered the student, not more for what the work furnishes, than for the sources of information which it points out, cannot fail to make it valuable. Indeed, it is only by the publication of such compends of the different branches of the science that we can hope, in the shortest time, and with the least labor, to become acquainted with their present limits, and the directions in which the student is most likely to be rewarded by new discoveries.

The work is divided into three parts; the first gives the theory of elimination between two equations; the second, between three equations; and the third, the elimination of many variables between any number of equations.

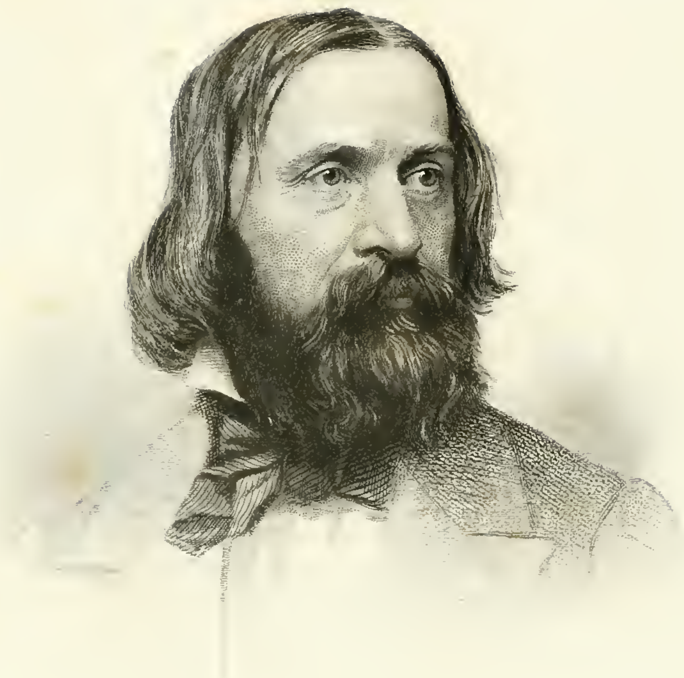
Editorial Items.

WE have received the following solutions of the Prize Problems in the March Number of the Monthly: —

AMANDA M. BENNETT, Saline, Mich., Probs. I., II.
HARRIET S. HAZELTINE, Worcester, Mass., Probs. I., II.
GEORGE A. OSBORNE, JR, Lawrence Scientific School, Cambridge, Mass., Probs. III., IV., V.
B. F. STANLY, Seminary, Cazenovia, N. Y., Prob. II.
JOHN J. CARTER, Nunda Lit. Institute, N. Y., Prob. I.
WILLIAM HINCHCLIFFE, Barre Plains, Mass., Probs. I., II., III.
FRANK N. DEVEREUX, Boston, Prob. II.
DAVID TROWBRIDGE, Perry City, N. Y., Probs. I., II., III., IV., V.
HORACE C. SYLVESTER, Boston, Prob. II.
JAMES F. ROBERSON, Ind. Univ., Bloomington, Probs. I., II., III.
ISAAC H. TURRELL, Mt. Carmel, Ind., Probs. I., II., V.
CHARLES FISH, Patten, Me., Prob. V.
ASHER B. EVANS, Madison Univ., N. Y., Probs. III., IV., V.
M. C. COREY, Exeter, N. H., Prob. II.

Errata. — On page 100, equations (10), for $q^{\frac{m}{n}}\ q^n$ read $\left(q^{\frac{m}{n}}\right)^n$; for $q^m \div q^n$ read $\sqrt[n]{q^m}$.

Books Received. — *Elements of Chemical Physics.* By JOSIAH P. COOKE, JR., Erving Professor of Chemistry and Mineralogy in Harvard University. 8vo. pp. 739. Boston: Little, Brown, and Co. 1860. *Théorie Nouvelle Géométrique et Mécanique des Lignes a Double Courbure,* par PAUL SERRET, Docteur ès Sciences, Membre de la Société Philomathique. Paris: Mallet-Bachelièr, Imprimeur-Libraire du Bureau des Longitudes, &c. 1860.



Benjamin Peirce

THE

MATHEMATICAL MONTHLY.

Vol. II. . . . JULY, 1860. . . . No. X.

PRIZE PROBLEMS FOR STUDENTS.

I. Solve the equations

$$x^2 y^2 (x^4 - y^4) = 2340,$$
$$x y (x y^3 - 1)(x^2 + y^2) = 1794,$$

by quadratics. — Communicated by E. A. Hopkins, Cleveland, Ohio.

II. Given the lengths of the three perpendiculars dropped from any point in the plane of an equilateral triangle upon the sides; to find the segments of the sides. — Communicated by F. E. Tower, Amherst College.

III. If e denote the edge of any regular dodecahedron, and $\beta = 36°$, prove that

$$\text{solidity} = \frac{5 e^3 \cot^2 \beta}{2 \sqrt{(4 \sin^2 \beta - 1)}} = \frac{5 e^3 \tan^2 45}{4 \sqrt{(\sin 6° \cos 24°)}}.$$

Also obtain similar formulas for the solidity of the icosahedron. — Communicated by Prof. D. W. Hoyt.

IV. A given cylindrical vessel, filled with water, is placed with its base upon a horizontal plane. It is required to determine the angle of inclination to which the plane must be raised before the vessel will fall, the water being at liberty to overflow its top. The base is supposed to be fixed so as to prevent it from sliding, but not from tilting when the plane is inclined. — Communicated by Professor Kirkwood.

V. Bisect the attraction which a sphere of varying density exerts

upon an exterior point; that is, divide the sphere so that the two parts shall exert the same attractive force in the same direction.

Solutions of these problems must be received by September 1, 1860.

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. VII., Vol. II.

The first Prize is awarded to John A. Winebrener, Princeton College, N. J.

The second Prize is awarded to George C. Round, Wesleyan University, Middletown, Ct.

The third Prize is awarded to Lewis Foote, O. C. Seminary, Cazenovia, N. Y.

Prize Solution of Problem I.

By Miss Harriet S. Hazeltine, Worcester, Mass.

Prove that an arithmetic mean is greater than a geometric.

Let $x - y$ and $x + y$ denote the extremes; then x is the arithmetic and $\sqrt{(x^2 - y^2)}$ the geometric mean, and it is evident that $x = \sqrt{x^2} > \sqrt{(x^2 - y^2)}$.

Second Solution.—Let $a > b$; then $a - b > 0$; $a^2 - 2ab + b^2 > 0$; $a^2 + 2ab + b^2 > 4ab$; $a + b > 2\sqrt{ab}$; $\therefore \frac{1}{2}(a + b) > \sqrt{ab}$. — G. S. Morison, Harvard College.

Prize Solution of Problem II.

Let three bodies with velocities V, V', V'', move uniformly in the same direction, in the circumference of a circle. Required the time of their conjunction, supposing them to quit a given point at the same time.

Let C denote the circumference of the circle; then, since $V - V'$ and $V - V''$ are respectively the gains of V upon V' and V'' in the same unit of time, $\frac{C}{V - V'}$ and $\frac{C}{V - V''}$ will denote the times which will elapse between the instant of starting and the conjunctions of V, V' and V, V'' respectively. And the least common multiple of

these times, or their product if they are prime to each other, will give the time which will elapse between successive conjunctions of the three bodies.

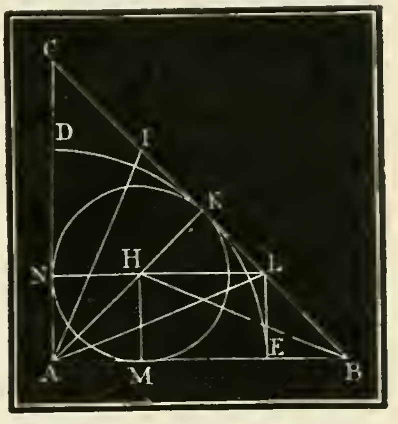
This is essentially the solution given by several of the competitors.

Prize Solution of Problem III.

By Stockwell Bettes, Boston, Mass.

The diameter of a circle inscribed in the quadrant of a second circle is equal to the side of the regular octagon circumscribed about the second circle.

Bisect the quadrant by the line AK, and draw the tangent CB at K. Next, bisect the angles CAK and KAB, and IL will be the side of the circumscribed octagon. The centre of the circle inscribed in the triangle CAB is the same as that inscribed in the quadrant, and it is at the intersection, H, of the lines bisecting the angles of the triangle. The triangles AKL and BKH are equal, and therefore $KH = KL$. But $KI = KL$; therefore $IL = 2\,HK$.



Second Solution. — LK and LE, tangents at K and E, are half sides of the octagon. Draw LN parallel to AB; then angle $KHL = KAB = KLH$, $\therefore KH = KL = LE = HM = HN$, therefore H is the centre of the inscribed circle. — John R. Emery, Princeton College, New Jersey.

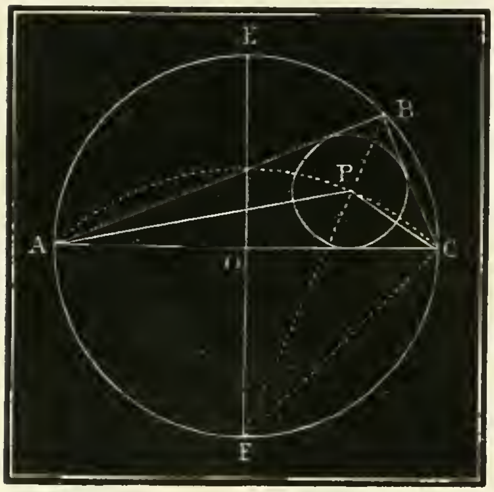
Prize Solution of Problem IV.

By Cadet Arthur H. Dutton, West Point, N. Y., and Hiram L. Gear, Marietta College, Ohio.

Required the locus of the centres of the circles inscribed within all the right-angled triangles which can be inscribed in a given semicircle.

Let P be the centre of a circle inscribed in any right-angled triangle, ABC, which can be inscribed in the semicircle AEB. Draw the diameter EF perpendicular to AC, and join BF, PC,

and $F\,C$. Because $B\,F$ bisects the angle $A\,B\,C$, it passes through the point P. But $F\,P\,C = P\,B\,C + B\,C\,P = A\,C\,F + P\,C\,A = P\,C\,F$.



Therefore the triangle $F\,P\,C$ is isosceles, and the point P is in the circumference of a circle of which F is the centre, and radius $F\,C = O\,C\sqrt{2}$.

Second Solution. — The centre of the inscribed circle is at the intersection of the lines bisecting the angles at the base, and as the sum of these angles is constant, the half sum is also constant; and hence the vertical angle of this second triangle is constant, and since the base is constant, the locus of these points must be in a circle. — John A. Winebrener, Princeton College, N. Y.

Third Solution. — Let (xy) be the co-ordinates of the point P, O being the origin, and r the radius of circle $A\,B\,C$. But $\tan(P\,A\,C + P\,C\,A) = \tan 45° = 1$, and since $\tan P\,A\,C = \frac{y}{r+x}$, $\tan P\,C\,A = \frac{y}{r-x}$, we have by the usual formula, $x^2 + y^2 + 2\,r\,y = r^2$; or changing the origin to F, $x^2 + y^2 = 2\,r^2$, therefore the locus of the point P is a circle whose centre is F.

All the analytical solutions are essentially the same.

Prize Solution of Problem V.

By W. F. Osborne, Wesleyan University, Middletown, Ct.

From a box containing a very large number of white and black balls, of each an equal number, three balls are taken at random and placed in a bag without being seen. A takes a ball at random from the bag, observes its color, and replaces it four times in succession. The ball was white on each of the four drawings. What are the respective probabilities that the bag contains 1, 2, or 3 white balls?

The balls as drawn from the box and placed in the bag will be, either no white, $b\,b\,b$; one white, $w\,b\,b$, $b\,w\,b$, $b\,b\,w$; two white, $w\,w\,b$,

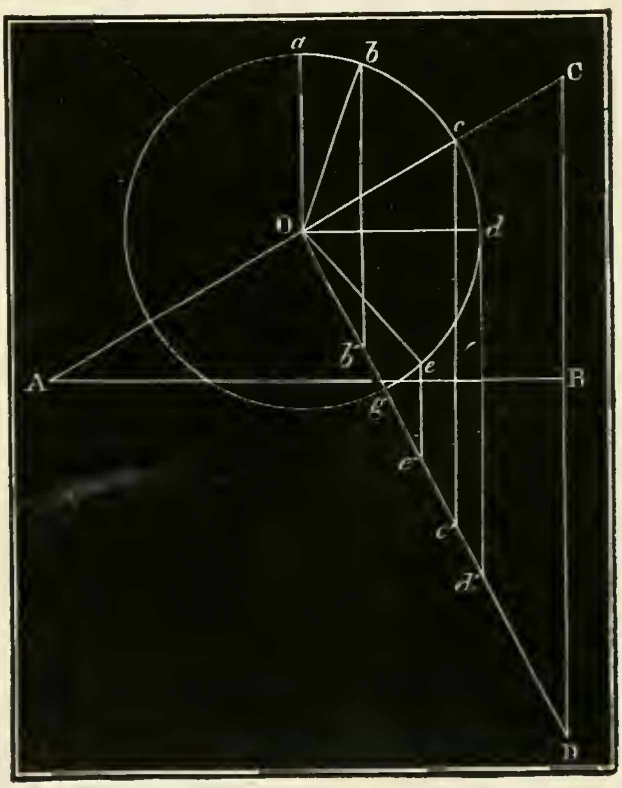
w b w, *b w w*; or three white, *w w w*. Hence, before drawing from the bag, the chances that the balls are one white, two white, three white are respectively $\frac{3}{8}$, $\frac{3}{8}$, $\frac{1}{8}$.

Now, if there is only one white ball in the bag, the chances that it will be drawn four successive times are $(\frac{1}{3})^4$; if two of the balls are white, the chances are $(\frac{2}{3})^4$; if three of the balls are white the chances are $(\frac{3}{3})^4$. Combining, we get $\frac{3}{8}(\frac{1}{3})^4 = \frac{1}{216}$, $\frac{3}{8}(\frac{2}{3})^4 = \frac{16}{216}$, $\frac{1}{8}(\frac{3}{3})^4 = \frac{27}{216}$, as the *a priori* probabilities that a white ball was drawn from the bag four times in succession. But since a white ball is drawn four successive times, the respective probabilities are

$$\frac{1}{1+16+27} = \frac{1}{44}, \quad \frac{16}{1+16+27} = \frac{16}{44}, \quad \frac{27}{1+16+27} = \frac{27}{44}.$$

NOTES AND QUERIES.

1. *Notes on the Inclined Plane and the Wedge.* — Let $A\,B\,C$ represent any inclined plane; W the weight, placed at O; P the power, which is constant and may be denoted by $O\,a$, the radius of a circle; and R the reaction in the line $O\,D$ perpendicular to the plane. The condition of equilibrium will be represented by the three sides of a triangle. R must always act in the line $O\,D$, W must be perpendicular to the horizon, and P may vary in direction, and will give different values for R and W in different positions.

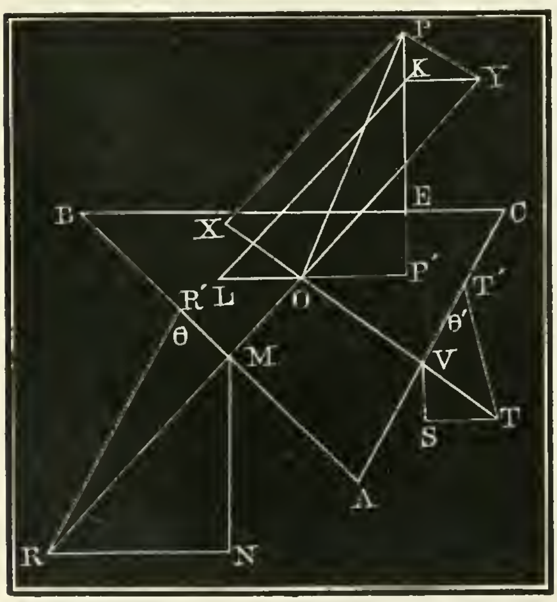


When P acts in the line $O\,a$,

W must be in the same line, and therefore $P = W$ and $R = 0$. When P acts in the line Og, R must act in the same line, and therefore $P = R$ and $W = 0$. All the positions including the limits are then represented as follows: —

$P = O\,a$	$W = O\,a$	and $R = 0$ (min.)
$P = O\,b$	$W = b\,b'$	" $R = O\,b'$
$P = O\,c$	$W = c\,c'$ (max.)	" $R = O\,c'$
$P = O\,d$	$W = d\,d'$	" $R = O\,d'$ (max.)
$P = O\,e$	$W = e\,e'$	" $R = O\,e'$
$P = O\,g$	$W = 0$ (min.)	" $R = O\,g$.

In general W increases from an equality with P to its maximum value when P acts parallel to the plane, and afterwards decreases again to its minimum value, or zero, when P acts perpendicular to the plane. R increases from its minimum value, or zero, to its maximum, when P acts parallel to the base of the plane, then decreases again to equality with P, when P acts perpendicularly to the plane. When P acts in directions beyond the limits $O\,a$ and $O\,g$, negative results for W and R are obtained, of an interesting character.



Let $A\,B\,C$ represent any wedge whatever; also $R\,R'$ and $T\,T'$ any resistances whatever, acting at any angles, θ and θ'. Resolve $R\,R'$ into $R\,M$ and $R'\,M$; and $R\,M$ into $R\,N$ and $M\,N$. Also in like manner resolve $T\,T'$ into $T\,V$ and $T'\,V$; and $T\,V$ into $V\,S$ and $S\,T$. Produce $R\,M$ and $T\,V$ until they meet in O, and beyond O take $O\,X = V\,T$, and $O\,Y = R\,M$. Next, complete the parallelogram $O\,X\,P\,Y$; then draw the diagonal $O\,P$, and lastly, resolve $O\,P$ into $O\,P'$ and $P\,P'$. $M\,N$ represents the effective force

of $R R'$, and $V S$ that of $T T'$. Therefore when there is an equilibrium, $M N + V S = P P'$.

From Y draw $Y K$ perpendicular to $P P'$, and complete the parallelogram $Y K L O$. The triangles $R M N$ and $L K P'$ are equal, since the sides are respectively parallel, and $L K = O Y = R M$ by construction; therefore $P' K = M N$. The triangles $P K Y$ and $T V S$ are equal, for a similar reason, and $P K = V S$. Consequently $M N + V S = P' K + K P = P P'$. This method of resolution also gives the place E, where the power $P P'$ must be applied to produce equilibrium. When the prolongation of $R M$ and $T V$ meet in a point O beyond $B C$, either to the right or left, negative results are obtained. — Prof. JOHN L. CAMPBELL, Wabash College, Crawfordsville, Ind.

2. *Law of Gravity.* Solution of the problem on page 204, Vol. II. — Two similar systems of bodies have all corresponding linear dimensions in the same ratio to each other; that is, the linear dimensions and the distances apart of the bodies are in the one system in a fixed ratio to those of the other.

In order that two similar systems may remain similar for successive instants of time, it is necessary that the motions of corresponding bodies be in the same relative directions *inter se*, and that the velocities of corresponding motions be proportional to the linear dimensions of the two systems.

If, now, these motions be continually modified by the action of central fixed forces, or by forces dependent directly upon the masses of the bodies, and upon some function of their distances apart, then, since the changes in the motions must also be proportional to the dimensions of the systems and to the motions themselves, the values of these central forces will be proportional to the same dimensions. But the forces, so far as they are dependent upon the masses, are proportional to the cubes of the linear dimensions; hence, so far as

they depend upon the distances, they must be inversely proportional to the squares of these dimensions, in order that on the whole, from the masses and distances combined, they may be simply dependent on the first power of the distances, dimensions, and velocities of the bodies. Hence the law of gravity, and conversely corresponding motions of revolution or oscillation in two similar systems must, by the law of gravity, have the same periods, since the dimensions of the paths or orbits of these motions, their velocities, and their changes of velocity, are all proportional to the dimensions of the systems.

Hence all measures of time, whether by periods of orbital, of rotary, or of oscillatory motions, are by the law of gravity independent of the dimensions of the material universe; and if the solar sytem had been constructed on the scale of a common planetarium, it would still have moved, by virtue of the forces inherent in matter, *pari passu*, through the same phases of motion and configuration, with the same periods as now. — W.

3. Develop the Naperian logarithm of x into a series. — Put $x = y + 1$; then $dx = dy$, and $\frac{dx}{x} = \frac{dy}{y+1}$. But by division

$$\frac{1}{y+1} = 1 - y + y^2 - y^3 + y^4 \text{ \&c.}$$

$$\therefore \frac{dx}{x} = \frac{dy}{y+1} = dy - y\,dy + y^2\,dy - y^3\,dy + y^4\,dy \text{ \&c.}$$

By integration

$$\log x = y - \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4 + \tfrac{1}{5}y^5 \text{ \&c.}$$

Restoring the value of $y = x - 1$, we get

$$\log x = (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4 \text{ \&c.}$$

— Artemas Martin, Franklin, Pa.

4. *Note on Right-angled Triangles.* — I have for many years kept on hand for my own convenience a list of the fifty two right-angled triangles described by Professor Hoyt in the May number of the Monthly. I prepared the series by using the formula

$$(a^2 + b^2)^2 = (a^2 - b^2)^2 + (2\,a\,b)^2,$$

which is easily seen to be true, and in which any numbers whatever may take the places of a and b. If for these letters we substitute the natural numbers in succession we shall obtain thirty sets of numbers no one of which will exceed a hundred; *fourteen* of these however are equimultiples of some of the others, and *twenty-two* other multiple sets may be found within the same limit. We thus find, as Professor Hoyt has done, *fifty-two* right-angled triangles whose sides are expressed by integral numbers not exceeding one hundred, and *sixteen* of which are dissimilar in form. I cannot now call to mind where I found the formula given above. — Prof. E. S. Snell, Amherst College, Mass.

5. *Note on Equal Temperaments.* — It is assumed that the number of vibrations in a given time, producing a musical tone, is to the number producing its octave as 1 is to 2; that the numbers in like manner corresponding to a note and its "fifth" are to each other as 1 to 1.5; and that in a scale of equal temperament the numbers corresponding to the successive tones are in geometrical progression. Required, the number of equal intervals into which an octave must be divided, so as to have one of the tones approximate nearly to a "fifth." Let $z =$ the ratio in the geometrical progression, $y =$ the number of intervals approximating the fifth. $x =$ the number of intervals in the octave. Then

(1) $z^y = 1.5$ nearly, (2) $z^x = 2$,

(3) $y \log z = \log 1.5$ nearly, (4) $x \log z = \log 2$.

Dividing (3) by (4) we obtain

$$\frac{y}{x} = \frac{176091}{301030} \text{ nearly,}$$

which, by the method of continued fractions, gives the approximate values

$$\frac{y}{x} = \frac{3}{5}, \frac{7}{12}, \frac{24}{41}, \frac{31}{53}, \frac{179}{306}, \text{\&c.}$$

The approximation $\frac{3}{5}$ gives $y = 3$, $x = 5$; whence by (2) $z = 1.14871$; and by (1) the fifth, $z^y = 1.557$. This error of .0157 is too great to be tolerated by the musical ear. The approximation $\frac{7}{12}$ would give $y = 7$, $x = 12$, $z = 1.05946$, and $z^y = 1.49830$. This is the usual chromatic scale, and as the error in the "fifth" is but .0017 it satisfies the ordinary ear. The approximation $\frac{24}{41}$ would give as the ratio of the fifth $z^y = 1.50042$, an error of only .00042. The approximation $\frac{31}{53}$ would, like $\frac{7}{12}$, give a flatted fifth; its ratio would be $z^y = 1{,}49994$, and this error of .00006 would probably be quite imperceptible even to the nicest ear. An instrument to play 53 notes in an octave would, however, probably be difficult of construction; nor can we expect voices to move with certainty through such small intervals. — M. H. Doolittle, Sophomore Class, Antioch College, Ohio.

Additional Note by Rev. Thomas Hill, President of Antioch College. — Mr. Doolittle has calculated the values of the approximation $\frac{3}{5}$ as follows: —

1.0000, 1.1487, 1.3195, 1.5157, 1.7411, 2.0000.

The nearest notes to these, in the twelve-semitone scale, are C, D, F, G, B-flat, C, which will instantly be recognized as the scale of B-flat, with the seventh and fourth omitted, that is, the scale of the old Scotch melodies. Thus it appears that this old-fashioned scale approximates rudely towards a division of the octave into five equal divisions. The simple fractions which approximate most nearly to these values are $\frac{8}{7}$, $\frac{4}{3}$, $\frac{3}{2}$, $\frac{7}{4}$, and perhaps the Scotch singer may follow these simpler divisions when singing without a keyed or fretted instrument. If, with Mr. Poole, we admit the prime seventh into our musical theories, this is a point perhaps worthy of investigation.

Mr. Doolittle's 5-note scale differs from the Scotch in taking C instead of B-flat as the tonic. If we add to it the note A, retaining C as the tonic, it becomes identical with the Irish scale, in which, for example, "Huggamur pene on Sambhruliun" is written.

THE MOTIONS OF FLUIDS AND SOLIDS RELATIVE TO THE EARTH'S SURFACE.

[Continued from Page 97.]

SECTION VI.

ON THE MOTIONS OF THE OCEAN.

72. Besides the actions of the sun and moon which give rise to the tides, there are only two causes which can produce any sensible motions on the waters of the ocean. One of these is the action of the atmosphere upon the surface of the ocean, and the other, the difference of density between the water near the equator and that towards the poles, arising from a difference of temperature. The general motions of the atmosphere at the surface of the ocean have a tendency to cause a westward motion of the water in the torrid zone, and an eastward motion in the middle and higher latitudes; and from what we know of the effects of strong winds upon the ocean, we have reason to think that these general motions of the atmosphere are adequate to produce *sensible* motions, since, after the inertia of the water is once overcome, which, however small the force, is only a question of time, the only force necessary is that which is adequate to overcome the resistance of friction, which is very small where the velocity is small. The difference of density between the equator and the poles causes a slight interchanging motion of the water between them, and consequently, where not interrupted by continents, it produces a system of motions in the ocean similar to those of the atmosphere. Hence these two causes

of oceanic disturbance, whatever their relative weight, both act in the same directions, and conjointly cause the observed westward motion of the ocean near the equator, and eastward motion towards the poles.

73. The westward motion of the water of the ocean in the torrid zone was first observed by Columbus, and is now well established; and observations also show that there is a motion towards the east in the higher latitudes. A bottle thrown into the ocean near Cape Horn was picked up three and a half years afterward at port Philip, Australia, a distance of 9000 miles, which makes the eastward velocity in that latitude more than 7 miles per day. And Sir James Ross, when sailing eastward near Prince Edward's Island, found himself every day from 12 to 16 miles by observation in advance of his reckoning. (*Voyage to the Antarctic Seas,* Vol. II. p. 96.) But a westward motion being established in the torrid zone, an eastward motion in the higher latitudes must be admitted; for, as was shown in the case of the atmosphere (§ 35), the one cannot exist without the other.

74. It has generally been supposed that the equatorial westward current of the ocean is caused principally by the action of the westward winds there; but Professor Guyot thinks that "it is too deep and rapid to admit of being explained by their action alone," and that "the difference of temperature between the regions near the equator and those near the poles controls all other causes by its power and the constancy of its action." (*Earth and Man,* pp. 189, 190.) The torsive or deflecting force which causes the westward motion of the atmosphere and the ocean in the equatorial regions, and the eastward motion in the higher latitudes, has been shown to be as the velocity of the interchanging motion between the equatorial and the polar regions; and hence if this motion in both were similar, the relative amount of this force in each must be as the

whole mass multiplied into the velocity of this motion between the equator and the poles. If we suppose the ocean to be 3 miles in depth, its mass is about 500 times that of the atmosphere, and hence if the motion between the equator and the poles were only $\frac{1}{500}$ of that of the atmosphere, the part of the force which gives it a westward motion near the equator, and an eastward motion toward the poles, arising from this cause, must be greater than that of the action of the atmosphere upon it, since the whole amount of this force in the atmosphere is not spent upon the ocean, but only that part which overcomes the resistances to its motions. Although the effect of temperature in producing a difference of density, and consequently of disturbing the equilibrium, is very much less in the ocean than in the atmosphere, yet since the amount of motion which a given disturbing force will produce where time is not considered, depends, as has been stated, upon the amount of the resistances, and not upon the amount of inertia to be overcome; and since the resistances diminish as the square of the velocity, a very small amount of disturbing force arising from a difference of density must be adequate to cause an interchanging motion in the ocean between the equatorial and the polar regions equal to $\frac{1}{500}$ of that of the atmosphere; and hence we have reason to think that a greater part of the motions of the ocean is due to this cause than to the action of the atmosphere upon it.

75. The motions of the ocean being similar to those of the atmosphere, they must cause a slight elevation of the surface about the parallels of 30°, and a depression at the equator and the poles, just as in the case of the atmosphere, except that it will be less in the ratio of the relative velocities of the motions of the ocean and of the atmosphere. If we suppose the east and west motions of the ocean to be $\frac{1}{60}$ of those of the atmosphere at the earth's surface, as given in the third column of the computed table (§ 48), which would

require the maximum eastward velocity in the southern hemisphere to be about 10 miles per day, it would cause the surface of the ocean in the southern hemisphere to be about 15 feet higher at the parallel of 30° than at the pole, and also a little higher than at the equator. Now if the motions which cause this accumulation of water were the same at the bottom of the ocean as at the surface, there would be no tendency of the water to flow out at the bottom from beneath this accumulation; but since the motions there must be much less, it must flow out both toward the equator and the pole, especially toward the latter, as the depression there is much the greater. Since the density of sea-water does not increase below the temperature of 28°, the density of the ocean does not increase beyond a certain latitude, and hence there is no flow of the water at the bottom from the poles toward the equator, arising from the maximum density at the pole, as seems to be the case in a very slight degree in the atmosphere, but the under current at the bottom, arising from the greater pressure about the parallel of 30°, must extend entirely to the poles; so that there must be a slight tendency of the water to rise at the poles, and flow at the surface some distance towards the middle latitudes. As the water toward the bottom of the ocean is always about the same as the mean temperature of the earth, when it first rises to the surface at the pole, it must be much warmer than it is after it has flowed some distance from it, and hence we have reason to think that there may be open polar seas, surrounded by barriers of ice at some distance from the pole, where there is the maximum temperature of the surface water. A surface current from the poles is indicated by the motions of icebergs in both hemispheres from the polar regions towards a lower latitude.

76. Where the east and west motions of the ocean are entirely intercepted by continents, as in the northern hemisphere, the water receives a slight gyratory motion from left to right. The westward

motion of the waters of the Atlantic in the torrid zone, impinging against the continent of America, causes the surface of the water of the Caribbean Sea and the Gulf of Mexico to be a little above the general level, while the eastward motion of the northern part of the Atlantic causes the surface of the water adjacent to the eastern coast of North America, in that latitude, to be a little lower. Hence there is a flow of warm water from the Gulf of Mexico along the coast of the United States toward the lower level about Newfoundland, which, on account of the peculiar configuration of the coast about the Gulf of Mexico and the peninsula of Florida, gives rise to the Gulf Stream. The eastward motion also of the northern part of the Atlantic causes the surface of the water on the western coast of Europe to be a little *higher* than the general level, while the westward motion in the torrid zone causes it to be depressed, on the western coast of Africa, a little *below* this level, and hence the water of the eastern side of the Atlantic, flowing from a higher to a lower level, has a motion toward the equator. The whole of the North Atlantic has therefore a very slight gyratory motion from left to right, and is supposed to make a complete gyration in about three years.

77. A portion of the equatorial current flowing from the higher level of the Caribbean Sea toward Cape Horn causes the Brazil current, which is deflected eastward by the general eastward motion of the Southern Ocean. The east side of the South Atlantic, as well as that of the North Atlantic, seems to have a motion toward the equator. Says Sir James Ross, "There is a current from the Cape of Good Hope along the west coast of Africa 60 miles wide, 200 fathoms deep, with a velocity of one mile per hour, of the mean temperature of the ocean." (*Voyage to the Southern Seas*, Vol. II. p. 35.) This cannot be a portion of the Mozambique current from the warm waters of the Indian Ocean, passing around the Cape of Good

Hope, and giving rise to the equatorial current of the Atlantic, as has been supposed, but must come from the colder waters of the Southern Ocean. Hence the South Atlantic also has a tendency to assume a gyratory motion, and the equatorial current of the Atlantic is merely the equatorial portion of these two gyrations, with perhaps a small part of the Mozambique current passing around the Cape.

78. The general eastward motion of the water of the northern part of the Atlantic, and the consequent depression of the water next the coast of North America, is the cause of the cold current of water flowing from Baffin's Bay and the east coast of Greenland, between the Gulf Stream and the coast of the United States, called the Greenland current. Since the warm water of the Gulf Stream, in flowing northward, is deflected toward the east (§ 32), and that of the Greenland current, in flowing south, tends toward the west, there is no intermingling of the waters of the two currents, but they are kept entirely separate as if divided by a wall, as has been established by the Coast Survey.

79. There must be a motion of the waters somewhat similar to the Gulf Stream and the Greenland current, wherever the great equatorial current impinges against a continent, and the eastward motion toward the poles is interrupted. Hence, on the eastern coast of South America there is the warm Brazil current which has been mentioned, and on the eastern coast of Asia there is the warm China current, flowing toward the north, similar to the Gulf Stream, and the cold Asiatic current insinuating itself between it and the coast, like the Greenland current. On the east coast of Africa, also, there is the Mozambique current flowing south like the Brazil current, and it is also now well established that, east of the Cape of Good Hope, the general tendency of the water is toward the south. This water must mingle with the general eastward current of the South Sea, and hence there is a slight tendency to a gyratory motion in the Indian Ocean also.

80. On the western sides of the continents there is a motion somewhat the reverse of this, and instead of a warm current flowing north, there is a cold one flowing toward the equator, as has been shown to be the case in the Atlantic. Hence, on the west coast of North America there is a flow of colder water along the coast from the north, and on the west coast of South America is Humboldt's current, much colder than the rest of the ocean in the same latitude, both tending toward the equator to join the great westward current there across the Pacific, and to fill up, as it were, the vacuum which this current has a tendency to leave about the equator, on the west coast of America.

81. When a portion of fluid on the earth's surface gyrates from left to right, the deflecting force arising from the earth's rotation being in this case toward the interior, the surface assumes a slightly convex form. If, however, the velocity of gyration were equal to twice that of the earth's rotation multiplied by the cosine of the polar distance, the centrifugal force arising from the gyration would be exactly equal to the centripetal force arising from the earth's rotation, and consequently they would neutralize each other, and if the velocity of gyration were still greater, the surface would be convex, as has been shown in § 30. The water of the North Atlantic having a very small gyratory velocity in comparison with that of the earth's rotation, the interior is a little elevated above the general level, and consequently the pressure upon the bottom increased. If we suppose a circular portion of it, 3,000 miles in diameter, with its centre on the parallel of 30°, to perform a gyration from left to right in three years, equation (50) would give an elevation of five feet in the middle above the level of the external part. This equation, however, on account of the term which has been neglected in the analysis (§ 25), is not strictly applicable to so large a portion of fluid, but still it gives the order of the effect produced. Now the

gyrations which cause this elevation in the middle being principally towards the top, the increased pressure upon the bottom causes the fluid there to flow out on all sides, with a very small velocity, towards the circumference, and hence the water at the surface has a slight tendency to flow in from all sides towards the interior to supply its place. This completely accounts for that vast accumulation of drift and sea-weed, covering a large portion of the interior of the North Atlantic, called the Sargasso Sea. From what has been stated, the North Pacific must also have a slight gyratory motion from left to right, and hence it likewise has its Sargasso Sea.

ON THE MATHEMATICAL THEORY OF HEAT IN EQUILIBRIUM.

By Simon Newcomb, Nautical Almanac Office, Cambridge, Mass.

SECTION I.

TEMPERATURE.

1. It is proposed to present the basis of this portion of the theory of heat in a simple form. We start from the following hypotheses which may be considered as inductions from observation.

First. Every material surface radiates heat at a rate dependent on the nature of its substance, and the amount of heat or caloric which it contains; and the law of direction of this radiation is such that the amount of heat which falls on any point is proportional to the solid angle subtended by the surface at that point.

Second. Every material surface absorbs a certain portion of the heat which radiates on it from other surfaces, transmits another portion, and reflects the remainder.

The term, material surface, is here used in nearly the same sense

as in mechanics, and is understood to mean a layer of matter so thin that whatever heat falls on one side of it will be equally absorbed by, or instantly diffused through, its whole thickness. A body may be considered as composed of an indefinite series of such surfaces, not in actual contact, and through which heat is conducted by successive radiation from particle to particle, or surface to surface. Since in our present paper we consider only heat in equilibrium, we shall not touch upon the laws of conduction, or molecular radiation, but shall suppose that the amount of heat which the external surface of a body radiates into the interior is just equal to that which it receives from it. In this case the temperature of the body is the same as that of its surface.

2. Two surfaces are said to be at the same temperature, when, being brought into contiguity, the amount of heat radiated from each is equal to that absorbed. One of these surfaces may be considered as that of a thermometer, and the temperature of the other will be measured by the temperature of the thermometer when the two are in equilibrium.

Let us now find the condition that two bodies shall be at the same temperature. For this purpose let φh represent the amount of heat radiated from a unit of surface of the first body in a unit of time, and $\varphi' h'$ that radiated from the second. h is supposed to represent the amount of heat contained in a unit of quantity of the body, and φ is a function dependent on the nature of the body. Also, let b and b' represent the fractions of the heat which the surfaces respectively absorb of the whole amount of heat which falls on them. When the heat is in equilibrium between the surfaces, it is evident that the quantity passing from the first surface to the second, by reflection and radiation, must be equal to that passing in the other direction. Represent this quantity by g. Then $b\,g$ will be the quantity absorbed by the first body in a unit of time, $b'g$ that

absorbed by the second. But the quantity absorbed being equal to that radiated, we have the two equations,

$$\varphi' h' = b' g, \qquad\qquad \varphi h = b g;$$

from which we obtain

$$\frac{\varphi h}{b} = \frac{\varphi' h'}{b'}, \tag{1}$$

and these quotients, or any function of them, we may take as the measure of the temperature of the bodies or surfaces in question. I say any function of them, because we can have no absolute measure of temperature, properly speaking. For example, what shall we take as the mean temperature between the freezing and boiling points of water, which we represent by 50° Centigrade, or 122° Fahrenheit? Suppose that we suffer water at the freezing point to absorb just half the heat necessary to raise it to the boiling point, and call its temperature 50° C. Suppose also that we take other substances at 0°, and suffer them to absorb half the heat necessary to raise them to 100°. If now it were found that all substances would under these circumstances exhibit the same temperature by exposure to the thermometer we should be justified in calling their temperature 50°. But such would be far from being the case; and mathematically considered, it is arbitrary what substance we shall select, the temperature of which should furnish our standard of 50°. But it is found that under the circumstances supposed, all *gases* would exhibit sensibly the same temperature, whether the experiment was performed on gas under constant pressure, or under constant volume, and moreover, in the former case the expansion would have been sensibly half of the whole expansion. It also seems highly probable that if we could subject solid substances to constant volume, we should find them to come into the same category with gases. Since, however, in the present paper, we consider only

equal temperatures, no difficulty will arise from the adoption of any arbitrary scale of temperature, we shall therefore take $\frac{\varphi h}{b}$ as the expression for the temperature of any body.

SECTION II.

SPECIFIC HEAT.

3. By the specific heat of a body is understood the amount of heat necessary to raise the temperature of a unit of its mass 1°. More properly it is the value of $\frac{dh}{d\tau}$, τ representing the temperature, and the mass of the body being supposed unity. The specific heat of a body is therefore, putting $\tau = \frac{\varphi h}{b}$,

$$\frac{b\,dh}{d\varphi h} = \varrho,$$

or *it is the expression of a relationship between the absorbent power of a body, and the differential of its radiant power.* It will be observed that we have supposed the absorbent power of the body to be constant, as its temperature rises; this is admissible because we cannot distinguish between the heat which a body reflects, and that which it radiates, so that even if the supposition of constant absorbent power be not true, all the phenomena will still be represented by suitably changing the form of the function φ.

To make the above definition of specific heat clear, suppose that $\frac{d\varphi h}{dh}$ is exceedingly small, in other words, that supplying the body with a large amount of heat will increase very slightly the amount of heat it radiates. Then the thermometer by being brought into contiguity with the body will need very little additional heat to make its increase of radiation compensate that of the body, and thus be brought into equilibrium with it, and thus a very large accession of heat to the body will make a very slight increase in its tempera-

ture. But it is evident that in this case the above expression for ϱ would be very great.

2d. Suppose now that b is exceedingly small, or that the surface reflects very nearly all the heat which falls on it, and that an increase of heat produces but an average increase in the rapidity with which heat is radiated by the surface. Then, if while the thermometer and the surface of the body are contiguous and in equilibrium, we add a small quantity of heat to the latter, the rapidity with which it radiates heat will be *slightly* increased. But, since nearly all the heat which is either radiated or reflected from the thermometer to the surface of the body is instantly reflected from that surface back to the thermometer, the latter may require a considerable addition of heat to make the additional heat absorbed by the surface equal to the additional amount radiated, and thus the addition of a small absolute amount of heat to the surface would have greatly increased its apparent temperature. But it is evident that in the case supposed the value of ϱ would be very small.

4. The fact, that if we expose a body to an external temperature slightly below its own temperature, the latter will fall just as fast as it would rise if the external temperature were elevated by the same amount above it, is sometimes adduced as indicating a relationship between the radiant and absorbent powers of surfaces in general. But from what has been said it is quite evident that no such relationship exists; moreover, the observed fact (as above expressed) is a necessary result of the hypotheses respecting the radiation and absorption of heat cited in the commencement of the present paper. For, let $\varphi' h'$ represent the radiant power, and b' the absorbent power of the external medium; g the rapidity with which heat passes from the body to the medium, g' the rapidity with which it passes in the opposite direction. We then have

$$g = \varphi h + (1 - b) g'; \qquad g' = \varphi' h' + (1 - b') g.$$

Also for the rapidity with which the body loses heat, we have

$$\frac{d h}{d t} = \varphi h - b g' = b b' \left(\frac{\varphi h}{b} - \frac{\varphi' h'}{b'}\right) = \frac{b b' (\tau - \tau')}{b + b' - b b'}.$$

If, then, the scale of temperature which we have adopted coincides with that of the experiment, the above-mentioned phenomenon will hold true for all differences of temperature; in any case it will be true when the differences are small.

SECTION III.

TEMPERATURE OF BODIES EXPOSED TO THE SUN.

5. Hitherto we have considered only the phenomena of heat in equilibrium between two contiguous surfaces. Let us now investigate the laws of temperature of bodies exposed only to the heat radiated from a distant centre, as the sun. Our first problem will be to find the temperature of a plane surface exposed perpendicularly to the rays of the sun, and backed by a perfect non-conducting surface, or by another surface in equilibrium with it. If we represent by r the intensity of the radiant heat of the sun, and by b_1 the coefficient of absorption of the surface for perpendicular rays, the condition that the surface shall radiate as much heat as it absorbs gives

$$\varphi h = b_1 r,$$

and for the temperature we have

$$\frac{\varphi h}{b} = \frac{b_1}{b} r.$$

A distinction is made between b_1 and b because a surface does not necessarily absorb an equal portion of heat falling on it at every angle. To find the relationship between b_1 and b, suppose a small extent of surface to be placed inside of a closed surface. If the apparent hemisphere formed by the portion of the closed surface seen from one side of the enclosed surface be divided into very nar-

row zones parallel to the latter, it follows from the first hypothesis that the amount of heat radiated on a spherical point on the enclosed surface will be proportional to the apparent area of the zone, and therefore to $\cos\theta$, when θ represents the angle which a line drawn from any point of a zone to the surface makes with the latter. But the amount of heat which falls on any assigned portion of the surface will be proportional to that which would fall on a spherical point multiplied by $\sin\theta$. Wherefore the quantity of heat which will fall on the surface, making angles of incidence between θ and $\theta + d\theta$, will be represented by $\varphi h \sin 2\theta\, d\theta$. If, then, b_θ represents the absorbent power of a surface for heat radiating on it at an angle θ, its average coefficient of absorption, or b, will be

$$\int_0^{\frac{1}{2}\pi} b_\theta \sin 2\theta\, d\theta = b.$$

If, now, b_θ is the same function of θ for all substances, and if the coefficient of absorption for the rays of the sun is the same as that for rays radiated by a thermometer, or bears the same ratio to it; then the surfaces of all substances ought, under the circumstances supposed, to attain the same temperature. It seems highly probable that this would be the case in nature.

The above expression for the number of rays which a surface, or any small portion of it, receives at any angle of incidence may be regarded as general. Suppose that a surface receives rays from a single source, and takes successively every possible angular position, in other words, that a normal to the surface points successively and equally in every direction of the celestial sphere. It is evident that the *time* spent by this normal at an angular distance of θ' from the radiating point will be proportional to $\sin\theta'$, and that the rapidity with which the surface receives rays, while the normal is at this angle, is proportional to $\cos\theta'$. Whence the number of rays which

it receives will, as before, be proportional to sin $2\,\theta'$, and sin $2\,\theta$, since we have $\theta' = 90° - \theta$.

6. Let there be a solid body, of any form whatever, but not concave in any part, and opaque to the rays of heat, exposed in the planetary spaces to the rays of the sun, supposed parallel, every side of the body being turned rapidly and indiscriminately toward the sun. To find its temperature, we observe that any element of its surface will, on the whole, receive one fourth as much heat from the sun as if it were exposed directly to its rays at right angles. The average quantity of heat radiated on a unit of the surface during a unit of time will then be $\frac{1}{4}\,r$. The average amount absorbed will be $\frac{1}{4}\,b\,r$. The amount radiated, $\varphi\,h$. Whence, as the condition of equilibrium, we have

$$\frac{\varphi\,h}{b} = \tau = \tfrac{1}{4}\,r.$$

And this we may regard as the average, or normal temperature due to the intensity of the rays of the sun.

7. We now come to a more important question. Suppose that the solid of the preceding paragraph is covered with a very thin layer, partially diathermanous to the rays of the sun, but entirely opaque to those emanating from the body. Let b' represent the coefficient of absorption of the layer with respect to the sun, b'' its coefficient with respect to the heat radiated from the enclosed body; b the coefficient of absorption of the body, supposed the same for all rays. Also, let r' represent the average amount of heat radiated directly from the sun, through the layer into the body in a unit of time, on a unit of surface of the body, $\varphi\,h$ and $\varphi'\,h'$ the radiant powers of the body and layer respectively. If, then, we represent, as before, by g the quantity of heat passing each way between the interior surface of the layer, and the exterior surface of the body, we have, in the case of equilibrium,

$$g = \varphi' h' + (1 - b'')\, g + r', \qquad g = (1 - b)\, g + \varphi h\,;$$

from which we obtain

$$\frac{\varphi h}{b} = \frac{\varphi' h' + r'}{b''}.$$

The condition of equilibrium between the heat radiated from the layer and that received from the sun gives

$$\varphi' h' = \frac{b' r}{4} + r'.$$

We have therefore for the temperature of the body

$$\tau = \frac{\varphi h}{b} = \frac{b' r + 8 r'}{4 b''}.$$

We conclude from this, that by covering a planet with an atmosphere, or other medium, having the property of glass, and probably of our own atmosphere, in being more diathermanous to the radiant heat of the sun than to that of the planet, and also having a very small absorbing power, the temperature of the planet might be raised to any limit. A similar remark will apply to a thermometer placed within a glass case, which ought to rise much higher, on exposure to the sun, than it would if placed in a similar case of rock-salt or wood. If, however, the atmosphere or the case have *not* the above-mentioned property, no possible combination of other properties can make it cause the planet to be above the normal temperature due to the radiant heat of the sun and stars. It is therefore a mistake to suppose that increasing the absorbent power of an atmosphere will increase its heating properties; on the contrary, it is evident from the last equation that such increase would *diminish* its power of raising the temperature of the planet above the normal. It would, however, tend to equalize the temperature at different times during the same day.

ANSWER TO PROF. F. W. BARDWELL'S NOTE "ON THE HORIZONTAL THRUST OF EMBANKMENTS." Vol. II., P. 52.

By Capt. D. P. Woodbury, U. S. Corps of Engineers.

Professor Bardwell has not I think fully considered the conditions of the problem in question. Equation (1), Vol. I., p. 176, is, I believe, "the formula usually given." It is given by Poncelet and by many others, and has often been extended to a similar problem, the sliding thrust of arches.

Place a book against a vertical wall and apply the horizontal force necessary to keep it there. What sustains the book? Nothing but the friction due to this horizontal force. Incline the wall, the weight of the book will introduce a new element of friction, but we are not at liberty to neglect the first. The whole effect of friction in retarding motion, is equal to the whole normal pressure multiplied by a constant.

The investigation in question was confined to a particular case, though a case of very frequent occurrence, not elsewhere, so far as I know, specially treated. I mean the case in which the surface of the embankment is parallel to the natural slope of earth.

Professor Bardwell had in view, probably, "the angle of friction." If that were the object sought, his formula would be correct; for then, by the conditions of the problem, the horizontal force would be zero.

If we take into consideration the cohesion of particles, c per unit of surface along the base of the prism tending to slide, equations (1), (2), (4), Vol. 1, p. 176, will be changed as follows: —

$$(1) \qquad F' \sin v + (F' \cos v + Q \sin v) f + \frac{c\,h \sin a}{\sin(a - v)} = Q \cos v.$$

$$(2) \qquad F' = Q \tan(a - v) - \frac{c\,h \sin^2 a}{\sin(a - v) \cos(a - v)}.$$

$$(4) \qquad F' = \tfrac{1}{2} h^2 \sin^2 a - \tfrac{1}{2} h \sin a \left(h \cos a \tan v' + \frac{2c \sin a}{\sin v' \cos v'}\right).$$

The maximum of F' will correspond to the least value of the subtracted quantity, that is to

$$\tan v' = \tan(a - v) = \sqrt{\frac{2c}{hf + 2c}};$$

and the maximum itself is

$$F = \tfrac{1}{2} h^2 \sin^2 a - h \sin a \cos a \sqrt{\frac{2ch}{f} + \frac{4c^2}{f^2}}$$
$$= \tfrac{1}{2} (dp)^2 - \frac{h \sqrt{(2cfh + 4c^2)}}{1 + f^2}.$$

Mathematical Monthly Notices.

A Treatise on Attractions, LAPLACE'S *Functions, and the Figure of the Earth.* By JOHN H. PRATT, M. A., Archdeacon of Calcutta, late Fellow of Gonville and Caius College, Cambridge, and author of "The Mathematical Principles of Mechanical Philosophy." pp. 126. Cambridge: MacMillan & Co., and 23 Henrietta Street, Covent Garden, London. 1860.

This little volume is really a monograph upon the Figure of the Earth, the subjects of Attractions and LAPLACE'S Functions being prefixed for the sake of giving, in the same volume, just what the student will need in the discussion. Short Treatises, on special subjects, like the one before us, and MR. GODFRAY'S on the Lunar Theory already briefly noticed in the Monthly, are becoming quite common in England, and the plan is a good one; for those who wish to study a particular subject can now, in many cases, find it in a small volume of moderate price. Besides, when a subject is treated by itself, in a single volume, the author is more likely to give it symmetry and completeness; and the attention of the student is not diverted by matter which does not belong to the subject. The first chapter gives the attraction of spherical and spheroidal bodies, and shows for what laws of attraction the matter may be considered as condensed into the centre, together with IVORY'S theorem for finding the attraction of an ellipsoid upon an external particle. The second chapter treats of LAPLACE'S Coefficients and Functions, giving the proof that

$$D_f^2 V + D_g^2 V + D_h^2 V = 0, \qquad \text{or } -4\varrho' \pi,$$

according as the attracted particle is or is not a part of the attracting mass, V being the *Potential.* This equation is shown to be true when R, the reciprocal of the distance of the attracted particle from any point in the body, is put in the place of V. The equation in R is then transformed into polar co-ordinates, and the method of expanding R into a series involving LAPLACE'S coefficients is given; and the remainder of the chapter is devoted to

the investigation of the properties of these coefficients. In chapter third LAPLACE'S coefficients are used to determine the attraction of bodies nearly spherical; and it is shown that the part of the potential V which pertains to the excess of the attracting mass over a sphere can be expressed in terms of these coefficients. The fourth chapter is devoted to the attraction of bodies neither spherical nor spheroidal, nor nearly so; and it is shown how to calculate the effects of high table lands, and irregular mountain masses, on the plumb-line or spirit-level. The next chapter, which is the first on the Figure of the Earth, investigates this figure considered as a fluid mass, and therefore consisting of nearly spherical strata; and shows that the surface of a homogeneous mass of fluid, in the form of a spheroid, revolving about an axis with uniform velocity, is in equilibrium. In this chapter we also find a review of MR. HOPKINS'S argument to show that the crust of the earth is at least one thousand miles thick; with some special investigations confirming this result. The second chapter investigates the figure on the sole hypothesis of the surface being one of equilibrium and nearly spherical; and is essentially the same as that of Professor STOKE'S Memoir Published in the *Cambridge Philosophical Transactions* for 1849. The third and last chapter shows how to determine the figure by geodetic operations. In the former chapter the ellipticity was found to be $\frac{1}{300}$ nearly, upon *a priori* grounds; and in the next and last chapter this result is tested by measurement, "by inquiring whether an ellipse can be found with its axis coinciding with that of the Earth, and cutting the plumb-line at stations along it at right angles; and whether the ellipticity of that ellipse is $\frac{1}{300}$." We commend this volume to those who wish to study the subject discussed in it.

Supplementary Researches in the Higher Algebra, by JAMES COCKLE, M. A., F. R. A. S. Read by Rev. ROBERT HARLEY, F. R. A. S., before the Literary and Philosophical Society of Manchester, England, Nov. 29th, 1859. (Abstract communicated to the Mathematical Monthly.)

"In these Supplementary Researches the author extends the elementary formulæ given in § 2 of his original memoir; compares the cyclical and the epimetric views of the function U; and, following the former, is led to a new cyclical theorem which affords an easy demonstration of a proposition asserted in § 28. Mr. COCKLE then applies Mr. HARLEY'S cyclical process to the deduction of certain relations between unsymmetric functions; relations attained with a facility which the labor Mr. COCKLE formerly expended upon epimetrics well enables him to appreciate. The author next considers his symbol θ as a rational and symmetric, but otherwise arbitrary, function of four other functions, one of the latter functions, again, being a rational, but otherwise arbitrary, function of four arbitrary symbols, and the remaining three functions being derived from it by the three phases of an interchange which, provided it be of the fourth degree, is otherwise arbitrary. He then expresses the results of all the binary interchanges that can be performed on θ in terms of the single ones (and it should be noticed that from these the results of the ternary and higher interchanges may be obtained), and infers that θ may be regarded as the root of a sextic of which the coefficients are symmetric functions of the four arbitrary symbols. Mr. COCKLE then shows that if we group the six forms of θ two and two, the two members of each group being derivable one from the other by the conjugate interchanges, then the members of a group are inseparable by any interchanges whatever that can be performed upon the arbitrary symbols which enter into θ. So that symmetric functions of symmetric groups may be formed which are unsymmetric in θ, but yet unchanged by any permutations of the four arbitrary symbols. Consequently, if we apply the four arbitrary symbols as multipliers to four of the roots of a quintic, add the products to the fifth root and make the sum a constituent of θ, the symmetric group-function will be a rational function of the fifth root, and therefore the root of a quintic into the coefficients of which the arbitrary

symbols enter symmetrically. In order to give the greatest simplicity to the sextic in θ, the arbitrary symbols may have any suitable values assigned to them, and if we strive after a SYMMETRIC PRODUCT we find that those values are unreal fifth roots of unity. Mr. COCKLE adds, that the method of Symmetric Products has no special affinity for any particular theory of equations, and that although the evanescence of the resolvent product brings it into relation with that of LAGRANGE and VANDERMONDE, yet that better results may be deduced by applying it to EULER'S and BEZOUT'S theory, and without supposing that product to vanish."

A Treatise on the Calculus of Finite Differences. By GEORGE BOOLE, D. C. L., Honorary Member of the Cambridge Philosophical Society, Professor of Mathematics in the Queen's University, Ireland. Cambridge: MacMillan & Co., and 23 Henrietta Street, Covent Garden, London. 1860. pp. 248.

This work is composed on precisely the same plan as the author's *Treatise on Differential Equations*, a notice of which may be found in a previous number of the Monthly. In the work before us, particular attention is paid to the connection of its methods with those of the Differential Calculus; and it possesses all those peculiar merits as a text-book which were found to characterize the Treatise on Differential Equations; namely, a natural and logical arrangement of the parts of the subject, each part being followed with a careful selection of appropriate examples, the answers to which are collected at the end of the volume. In the brief space at our disposal, we cannot do better than give the heads under which the subject is treated. Chapter I. Nature of the Calculus of Finite Differences; II. Direct Theorems of Finite Differences; III. Of Interpolation; IV. Finite Integration; V. Convergency and Divergency of Series; VI. The Approximate Summation of Series; VII. Equations of Differences; VIII. Equations of Differences of the first Order, but not of the first Degree; IX. Linear Equations with variable Coefficients; X. Of Equations of Partial and of Mixed Differences, and of Simultaneous Equations of Differences; XI. Of the Calculus of Functions; XII. Geometrical Applications.

This summary must suffice to give a general idea of the work, which we especially recommend to those teachers and students who wish to have at least one good work upon each of the various departments of mathematics.

Astronomical Notices, No. 20. Albany, June 30, 1860. — Besides a very complete and valuable article on the Solar Eclipse of July 18, 1860, by R. T. PAINE, ESQ., of Boston, we find a letter from PROF. G. P. BOND, from which we extract elements of the Comet discovered at Harvard College Observatory, June 21, by MR. H. P. TUTTLE.

Elements of Comet III., 1860.

H. P. TUTTLE.	T. H. SAFFORD.
$T = 1860$, June 15.76914, Gr. m. t.	$T = 1860$, June 15.4618, Wash. m. t.
$\log q = 9.46238$	$\log q = 9.45862$
$\pi = 160^\circ\ 34'\ 53''$ Ap. Eq.	$\pi = 160^\circ\ 31'\ 35''$ Ap. Eq.
$\Omega = 84\ 48\ 15$ Ap. Eq.	$\Omega = 85\ 10\ 31$ Ap. Eq.
$i = 79\ 19\ 5$ Ap. Eq. Motion direct.	$i = 79\ 20\ 41$ Ap. Eq.

These orbits are computed from Cambridge observations of June 21, 24, 25.

Editorial Items.

We have received the following solutions of the Prize problems in the April number of the Monthly: —

Harriet S. Hazeltine, Worcester, Mass., Probs. I., II.
R. B. Canfield, Columbia College, N. Y., Probs. I., III., IV.
Isaac H. Turrell, Mt. Carmel, Ind., Probs. III., IV., V.
Henry B. Waterman, Yale College, Ct., Probs. I., II., III., IV., V.
E. O. Gibson, Sheshequim, Pa., Probs. I., II., III.
D. G. Bingham, Ellicotteville, N. Y., Probs. III., IV.
William Minto, University of Michigan, Probs. III., IV.
John R. Emery, College of New Jersey, Probs. III., IV., V.
Cadet Arthur H. Dutton, Military Academy, West Point, Probs. I., II., III., IV., V.
Charles Fish, Patten, Me., I., II., III., IV.
George B. Hicks, Cleveland, Ohio, Probs. III., IV., V.
Gustavus Frankenstein, Springfield, Ohio, Probs. III., IV.
John A. Winebrener, Princeton College, N. J., Probs. III., IV., V.
William Hinchcliffe, Barre Plains, Mass., Probs. I., III., IV.
Horace C. Sylvester, Boston, Mass., Prob. III.
F. E. Tower, Amherst College, Mass., Probs. III., IV.
Levi S. Packard, Chatham, N. Y., Probs. I., II., III.
S. J. Baldwin, Chester, N. J., Probs. I., II.
Lewis Foote, Cazenovia Seminary, N. Y., Probs. I., II.
William Simpson, White Plains, N. Y., Probs. I., III., IV.
W. F. Osborne, Wesleyan University, Ct., Probs. III., IV., V.
M. K. Bosworth, Marietta College, Ohio, Probs. III., IV., V.
Hiram L. Gear, Marietta College, Ohio, Probs. III., IV., V.
G. S. Morison, Harvard College, Prob. I.
Asher B. Evans, Madison University, N. Y., Probs. I., II., III., IV., V.
George A. Osborne, Jr., Lawrence Scientific School, Cambridge, Mass., Probs. III., IV., V.
George C. Round, Wesleyan University, Probs. III., IV., V.
W. W. Wetmore, Hamilton College, N. Y.. Probs. III., IV., V.
David Trowbridge, Perry City, N. Y., Probs. I., II., III., IV., V.
H. C. Corey, Exeter, N. H., Prob. III.
Stockwell Bettes, Boston, Mass., Probs. III., IV., V.

Harvard Mathematical Prizes. — Two Prizes, of two hundred and fifty dollars each, offered by the Hon. John C. Gray, "to the two members of the Class of 1860 who shall be found, after a special and thorough examination in the Second Term of their Senior year, to have made the greatest proficiency in the study of Pure Mathematics," have been awarded to C. M. Woodward, of Fitchburg, Mass., and C. A. Phillips, of Salem, Mass., by a committee consisting of Mr. J. B. Henck, Dr. B. A. Gould, Mr. J. D. Runkle, and Mr. Chauncey Wright, in connection with the Instructors in the Mathematical department of the College. Mr. Gray has offered the same Prizes, with the same conditions, to members of the Class of 1861.

Boyden Premium. — U. A. Boyden, Esq , of Boston, Mass., offers a Premium of one thousand dollars "to any resident of North America, who shall determine by experiment whether all rays of light and other physical rays, are, or are not transmitted with the same velocity." Competitors must transmit their memoirs to William Hamilton, Actuary of the Franklin Institute, Philadelphia, before the first of January, 1862.

Prof. Peirce's Portrait. — It gives us great pleasure to present to our readers in this number of the Monthly an admirable portrait of Prof. Benjamin Peirce, of Harvard University, engraved by the eminently successful and distinguished artist, H. Wright Smith, Esq., of Boston, from an excellent daguerrotype taken by Messrs. Southworth and Hawes; and to be able to assure all his personal and scientific friends, both at home and abroad, that it is recognized by his family and intimate friends as a most accurate likeness. The daguerreotype was taken just previous to his sailing for Europe, and we can see in the portrait a faint trace of the ill health which it is hoped his six months' absence abroad will entirely remove.

Prof. William Ferrel and Mr. Simon Newcomb, of the Nautical Almanac Office, have been detailed by the Superintendent, Commander C. H. Davis, to observe the Solar Eclipse of July 18th, at Cumberland House, a station of the Hon. Hudson's Bay Company on Saskatchewan River.

Edward Sawyer, Esq., of Boston, sends us the following Errata in Shortrede's Logarithmic Tables. On page 7, log 3262, for 3.2134840 read 3.5134840; on page 31, log 24451, for .3882996 read .3882966. On page 285, line 2, Mathematical Monthly, Vol. II., for "*acute angles*," read adjacent angles; on page 286, line 9, for $\frac{1}{\sin^2 \omega}$ read $\frac{1}{\sin^2 \alpha'}$.

Books Received. — *Illustrated Catalogue of Philosophical Apparatus.* Edward S. Ritchie, No. 313 Washington Street, Boston. This new edition (1860) of 84 pages, on tinted paper, well printed, and elegant in all respects, contains letters from twenty-six distinguished physicists in different parts of the country who are using Mr. Ritchie's Apparatus with entire satisfaction. These letters bear ample testimony of mechanical skill, and Mr. Ritchie's recent election to a Fellowship in the American Academy of Arts and Sciences attests the scientific ability and acquirements with which this skill is directed. *A Treatise on the Calculus of Finite Differences.* — By George Boole, D. C. L., Honorary Member of the Cambridge Philosophical Society, Professor of Mathematics in the Queen's University, Ireland. Cambridge: MacMillan & Co., and 23 Henrietta Street, Covent Garden, London. 1860. pp. 248. *A Treatise on Plane Co-ordinate Geometry* as applied to the Straight-Line and the Conic Sections. With numerous Examples. By I. Todhunter, M. A., Fellow and Assistant Tutor of St. John's College, Cambridge. Second edition revised. Cambridge: MacMillan & Co. 1858. pp. 316. *Elements of English Composition.* — Grammatical, Rhetorical, Logical, and Practical. Prepared for Academies and Schools, by James R. Boyd, A. M. New York: A. S. Barnes and Burr, 51 and 53 John Street. 1860. *Class-Book of Botany.* — Being Outlines of the Structure, Physiology, and Classification of Plants. With a Flora of all parts of the United States and Canada. By Alphonso Wood, A. M. New York: A. S. Barnes and Burr. 1860. *Manual of Geology:* — Designed for the use of Colleges and Academies. By Ebenezer Emmons, State Geologist of North Carolina. Illustrated with numerous engravings. Second edition. New York: A. S. Barnes and Burr. 1860. *Elements of Analytical Geometry and the Differential and Integral Calculus.* — By Charles Davies, LL. D., Professor of Higher Mathematics, Columbia College, N. Y. 8vo. pp. 398. New York: A. S. Barnes and Burr. 1860.

THE

MATHEMATICAL MONTHLY.

Vol. II. . . . AUGUST, 1860. . . . No. XI.

PRIZE PROBLEMS FOR STUDENTS.

I. PROVE that an equation of the second degree cannot have more than two roots.

II. The sum of n numbers in arithmetical progression is a, and the sum of their squares is b. Required, the first term and the common difference. — Communicated by ARTEMAS MARTIN, Franklin, Pa.

III. Given, three lines, a, b, c, drawn from any point within a regular polygon of n sides to any three of its consecutive corners, A, B, C. Required, the side of the polygon, and also a geometrical solution, granting that a polygon of n sides can be constructed when one side is given. — Communicated by PROF. C. A. YOUNG, Western Reserve College, Hudson, Ohio.

IV. Deduce the formulæ of Arithmetical and Geometrical Progressions by the method of differences. — Communicated by PROF. HENRY H. WHITE, Kentucky University, Harrodsburg.

V. If three points on the surface of a sphere (radius $= 1$) be joined by three arcs of small circles whose angular radii are $\varrho, \varrho', \varrho''$, the area of the included triangle will be

$$\theta + \theta' + \theta'' - \pi \pm \varphi \cot \varrho \pm \varphi' \cot \varrho' \pm \varphi'' \cot \varrho'';$$

in which $\varphi, \varphi', \varphi''$ are the lengths of these arcs, and $\theta, \theta', \theta''$, the angles at which they intersect. — Communicated by ASHER B. EVANS.

Solutions of these problems must be received by October 1. 1860.

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. VIII., Vol. II.

The first Prize is awarded to G. B. Hicks, Cleveland, Ohio.

The second prize is awarded to Asher B. Evans, Hamilton, N. Y.

The third prize is awarded to J. G. Weinberger, Millersville, Pa.

Prize Solution of Problem I.

By J. G. Weinberger, Normal School, Millersville, Pa.

I. If x is a whole number, prove that $x^3 - x$ is always divisible by 6.

Since $x^3 - x = (x - 1)\, x\, (x + 1)$, it appears that when x is a whole number, the given expression is the product of three consecutive whole numbers. But we know that at least one of three successive whole numbers is divisible by 2, because at least one must be even; and also, that one of them must be divisible by 3, because if the first one is not divisible there must be a remainder of either 1 or 2; and hence it follows that either the second or third number will be divisible. Therefore $x^3 - x$ is divisible by $2 \,.\, 3 = 6$.

Prize Solution of Problem II.

By D. G. Bingham, Ellicotteville, N. Y.

II. Describe a series of circles whose areas shall be one half, one fourth, one eighth, &c. that of a given circle.

Draw from C, the centre of the given circle, two radii, CA and CB, perpendicular to each other, and draw AB, the chord of 90°. From C drop a perpendicular, CK, upon this chord, and with C as a centre, and radius CK, describe a circle; then, as is easily seen,

$$(1) \qquad CA^2 : CK^2 = 2 : 1.$$

But since the areas of two circles are to each other as the squares of their radii, it follows from (1) that the area of the circle CK is half the circle CA. In precisely the same way we can find a circle

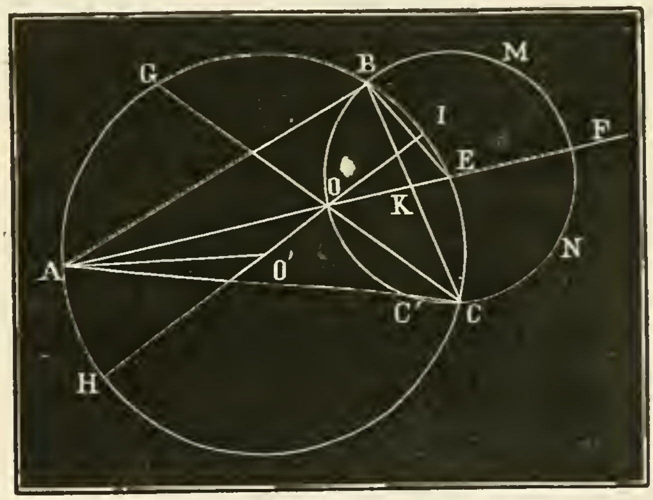
which is half of the circle $C K$, and therefore one fourth of circle $C A$. The process may be continued as far as we please.

PRIZE SOLUTION OF PROBLEM III.

By GEORGE B. HICKS, Cleveland, Ohio.

III. If R and r be the radii of the circles circumscribing and inscribing any triangle, and D the distance between their centres, then $D^2 = R^2 - 2 R r$. Required, a geometrical demonstration.

Let O' be the centre of the circumscribed, and O of the inscribed circles. Produce $A O$ indefinitely, and $O O'$ to meet the circumscribed circle in H and I. Draw $C G$, $C E$, $E B$. Lastly, from E as centre, with radius $E O$, describe the circle $O N F M$. This circle will pass through B and C. For $E C O$ is measured by $\frac{1}{2}(G B + B E)$ and $E O C$ by $\frac{1}{2}(A G + E C)$. But $G B + B E = A G + E C$. Hence the triangle $E O C$ is isosceles, and $E C = O E$. Similarly $B E = O E$. Also, since $A E$ bisects $B A C$, we have



$$B K = \frac{A B \times B C}{A B + B C}.$$

The chords $H I$, $A E$ give

$$H O \times O I = (R + D)(R - D) = A O \times O E.$$

But it is shown in Geometry that $(R + D)(R - D) = R^2 - D^2$;

$$\therefore D^2 = R^2 - A O \times O E.$$

It remains to show that $A O \times O E = 2 R r$.

The similar triangles $A B K$, $A E C$ give $\dfrac{O E}{A O + O E} = \dfrac{B K}{A B}$;

$$(1) \quad \therefore \frac{OE}{AO + 2\,OE} = \frac{BK}{AB + BK}, \quad \text{or} \quad \frac{OE}{AF} = \frac{BC}{AB + AC + BC}.$$

And from the secants AF, AC, $AO \times AF = AC' \times AC$. But since E is on the bisector of the angle BAC, $AC' = AB$;

$$(2) \qquad \therefore AO \times AF = AB \times AC.$$

Multiply (1) and (2), and $AO \times OE = \dfrac{AB \times AC \times BC}{AB + AC + BC}$. But $r(AB + AC + BC) = \dfrac{AB \times AC \times BC}{2\,R}$, since each side of this expression represents double the area of ABC;

$$\therefore \frac{AB \times AC \times BC}{AB + AC + BC} = AO \times OE = 2\,Rr,$$

as required.

Prize Solution of Problem IV.

By George B. Hicks, Cleveland, Ohio.

IV. If a circle cut a conic section in four points, A, B, C, D, and a second circle cut the same conic section in A, B, E, F, then will CD and EF be parallel.

Let $y^2 - mx - nx^2 = S = 0$ be the equation of any conic, and $x \cos a + y \sin a - p = \alpha = 0$, $\quad x \cos b + y \sin b - p = \beta = 0$ (1), the equations of the chords AB, CD.

Then will $S + k\,\alpha\,\beta = 0$ (2) be the equation of *some figure* through the intersections of (1) with the conic, the species of (2) being determined by the value attributed to the arbitrary constant k; for (2) is satisfied by $S = 0$, $\alpha = 0$; $S = 0$, $\beta = 0$, and therefore passes through the intersections of S, α, and β. (Salmon's Conic Sections, p. 211.)

To determine k so that (2) may represent a circle, write for S, α, and β their values at full length, equate the coefficients of x^2 and y^2, and equate to zero the cofficient of $x\,y$. We get

$$1 + n = k \cos(a + b), \qquad k \sin(a + b) = 0.$$

Squaring and adding,

$$(1+n)^2 = K^2 \quad \therefore \quad K = n + 1.$$

And we must therefore have

$$\cos(a+b) = 1, \qquad \sin(a+b) = 0,$$

$$\therefore \quad a + b = 0.$$

Now a and b are the angles made with the axis of x by the perpendiculars from the origin on the lines AB, CD.

Hence AB and CD are equally inclined to the axis of the conic (this being the axis of x). So also are AB, EF, since the property just proved is evidently general.

And therefore CD, EF make equal angles with the axis; $\therefore$ they are *parallel*.

Mr. Trowbridge adds to his solution of this problem the following:

"If two conics of the same kind cut a third conic in the points A, B, C, D and A, B, E, F respectively, then will CD and EF be parallel, provided the transverse axes of the cutting conics are either parallel or perpendicular to each other."

Prize Solution of Problem V.

By Asher B. Evans, Madison University, Hamilton, N. Y.

V. In any conic section, let u denote the length of the perpendicular dropped from any point in the curve upon the directrix, r the distance of this point from the nearest focus, a and b the semi-axes; prove that

$$\frac{u^2 - r^2}{u^2} = \frac{b^2}{a^2}.$$

A conic section is the locus of a point whose distances from a given fixed point, called the focus, and a straight line given in position, called the directrix, are always to one another in a constant ratio. This condition gives $r : u :: e : 1$, or $r^2 = u^2 e^2$, where e is the eccentricity of the conic section. Therefore $\frac{u^2 - r^2}{u^2} = 1 - e^2 = \frac{b^2}{a^2}$ for the ellipse, $-\frac{b^2}{a^2}$ for the hyperbola, and zero for the parabola.

NOTES AND QUERIES.

1. *Problems in Division, in which all the figures in the divisor but the right-hand one are nines.* — Let it be required to divide 864835 by 996. Proceed as follows: —

$$\begin{aligned} 864835 = 864 \times 1000 + 835 &= 864\,(996 + 4) + 835, \\ &= 864 \times 996 + 864 \times 4 + 835, \\ &= 864 \times 996 + 3456 + 835. \end{aligned}$$

Hence 996 is contained in 864835, 864 times, and 3456 + 835 remainder. But

$$3456 = 3\,(996 + 4) + 456 = 3 \times 996 + 3 \times 4 + 456\,;$$

hence 3456 contains 996, 3 times, with $3 \times 4 + 456 = 468$ remainder. Hence, 864835 contains 996 $864 + 3$ times, with $835 + 468 = 1303$ remainder. But

$$\begin{aligned} 1303 = 1\,(996 + 4) + 303 &= 1 \times 996 + 4 + 303, \\ &= 1 \times 996 + 307\,; \end{aligned}$$

that is, 1303 contains 996 once, with 307 remainder; and the final result is 868 as the quotient, and 307 remainder.

We can now condense the work into the following

OPERATION.

$$\begin{array}{r|l}
864 & 835 \\
3 & 456 = 864 \times 4 \\
 & 12 = 3 \times 4 \\
\hline
867 & = \text{quotient} \\
1 & 303 = \text{remainder} \\
 & 4 = 3 \times 4 \\
\hline
868 & = \text{quotient} \\
 & 307 = \text{remainder}
\end{array}$$

EXPLANATION.

Cut off as many figures from the right of the dividend as there are figures in the divisor. Multiply the remaining figures of the dividend, 864, by 4, the complement of the divisor, and set the product underneath the dividend. All the figures of this product over three, the number in the divisor, multiply by 4, and set the product underneath; and so continue to do until none fall to the left of the vertical line. Next add, and all at the left of the vertical line, as 867, will be quo-

tient; and all to the right, as 1307, will be remainder. Treat this remainder in the same way, and so on; and the last additions will give the true quotient and remainder. — Prof. DEVOLSON WOOD, University of Michigan, Ann Arbor.

2. *Derivative of* a^x. — Let $u = a^x$; then $\log u = x \log a$. By algebra the Naperian log of u is

$$\log u = \log \frac{z+1}{z} = 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \&c.\right),$$

$$= 2\left\{\frac{u-1}{u+1} + \tfrac{1}{3}\left(\frac{u-1}{u+1}\right)^3 + \tfrac{1}{5}\left(\frac{u-1}{u+1}\right)^5 + \&c.\right\}$$

since, if $u = \frac{z+1}{z}$, then $\frac{1}{2z+1} = \frac{u-1}{u+1}$.

By differentiating, we get

$$4\,du\left\{\frac{1}{(u+1)^2} + \frac{(u-1)^2}{(u+1)^4} + \frac{(u-1)^4}{(u+1)^6} + \frac{(u-1)^6}{(u+1)^8} + \&c.\right\} = dx \log a.$$

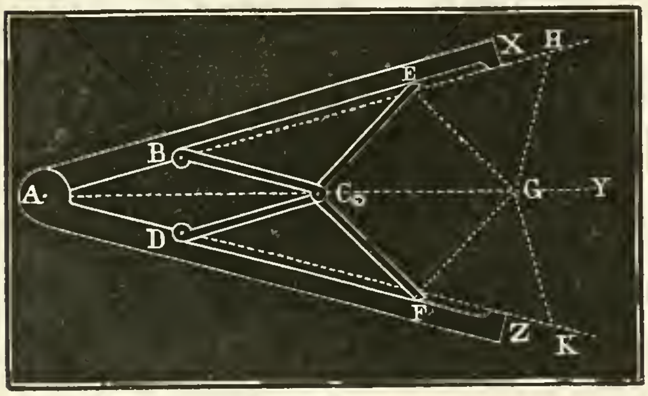
The series in the parenthesis is converging for all positive values of u, and by summing an infinite number of terms the equation becomes

$$4\,du\left(\frac{1}{4u}\right) = dx \log a. \quad \therefore \quad \frac{du}{dx} = a^x \log a.$$

— LUCIUS BROWN, Fall River, Mass.

3. *Remarks upon a (supposed) new Instrument for the Mechanical Trisection of an Angle.* — The *Philosophical Magazine* for April, 1860, contains an account, by Mr. TATE, of an instrument for the mechanical trisection of an angle, which is there described as new. The instrument is simple and elegant, and very convenient for the purpose intended. It will be seen, however, that the construction in question has long been known; for on turning to the 252d page of the second edition of the *Traité Analytique des Sections Coniques*, by DE L'HÔPITAL, first published in Paris in the year 1707, and republished in 1776, we

find the description of an instrument designed for the same object, and which, in its essential parts, is precisely the same as that mentioned above. It is somewhat singular that this should thus have escaped notice, especially as the name of De l'Hôpital is by no means an obscure one, and the problem of the trisection of an angle is one of so great historical celebrity. As the subject is one of some interest, we will give a brief description of the instrument employed by De l'Hôpital. AX and AZ are two rulers united by a joint at A. CB, CD, CE, and CF are four others, all equal, and united together by a pin at C, so as to turn freely. CB and CD are likewise connected with AX and AZ by joints at B and D in such a manner that $AB = BC$, and $AD = DC$, so that $ABCD$ is always a parallelogram, the ends E and F of CE and CF sliding easily upon AX and AZ respectively. Through A and C draw ACY. The $\angle ECY = \angle AEC + \angle EAC$. But



$$AEC = \angle EBC = \angle BAC + \angle BCA = 2 \angle EAC.$$

$$\therefore\ ECY = 3 \angle EAC.$$

Similarly, $\angle FCY = 3 \angle FAC$. $\therefore\ \angle ECF = 3 \angle EAF$.

Therefore, in order to trisect any angle, we have only to place the instrument in such a position that the sides of the rulers CE and CF coincide with the lines forming the given angle; in which case the straight lines drawn through C, parallel to AX and AZ, will divide it into three equal parts. De l'Hôpital has also shown (as is indicated by the dotted lines in the figure) how, by the addition of four rods, each equal to those meeting at C, and connected to each other

in a similar manner at G, as also to CE, CF, AX, and AZ at the points E, F, H, and K respectively, any angle may be divided into five equal parts, and that the construction may be extended in the same way, so as to effect the division of an angle into any odd number of equal parts. — Arthur W. Wright, New Haven, Ct.

4. *Note on the Irreducible Case.* — The following is a convenient method for approximating to a root of an equation of the form $x^3 - ax = b$.

Dividing by x, transposing, &c., we have

$$x = \pm\sqrt{a + \frac{b}{x}}.$$

Therefore

$$x = \pm\sqrt{a + \cfrac{b}{\pm\sqrt{a + \cfrac{b}{\pm\sqrt{a + \cfrac{b}{\&c.}}}}}}$$

Now, in order to find an approximate value of this continued fraction, take a few of its terms, and, beginning with the last, find the square root of $a + b$, divide b by this root, then add a to the quotient, extract the square root again, and continue the same process upward. The values of the radicals, thus successively found (which we may denote by R, R', R'', &c.), will be nearer and nearer approximations to the value of x. Considering all the radicals as positive, we shall thus find a positive root of the equation. It may then be depressed to a quadratic. The work may be somewhat abbreviated by the use of logarithms. In the following solution all the figures are retained.

$$x^3 - 6x = 2.$$

$$x = \pm\sqrt{6 + \cfrac{2}{\pm\sqrt{6 + \cfrac{2}{\&c.}}}}$$

Log. 2	0.301030	
" 8	0.903090(2	
	0.451545	log. R
	$\bar{1}$.849485	" .7071
" 6.7071	0.826535(2	
	0.413267	" R'
	$\bar{1}$.887763	" .77226
" 6.77226	0.830734(2	
	0.415367	" R''
	$\bar{1}$.885663	" .76853
" 6.76853	0.830494(2	
	0.415247	" $R''' = 2.6016$

Here, after four approximations, a value of x is correctly found to four places of decimals. In the equation $x^3 - 3x = 1$, we shall in the same manner find, after four approximations, $x = 1.8793$, which is also correct to four places of decimals.

An equation of the form $x^3 - ax = -b$ reduces to the form

$$x = \pm \sqrt{a - \cfrac{b}{\pm \sqrt{a - \cfrac{b}{\&c.}}}}$$

In this case, if we consider each radical as negative, the second term under each radical will be additive; and the process will be exactly the same as before, and the approximations as rapid. Thus in the equation $x^3 - 7x = -7$, the fourth approximation gives $x = -3.048$, which is correct to every figure.

This method, which is new to the writer, applies to the irreducible case of Cardan, of which the foregoing equations are examples. It serves to find a value of x "to any required degree of correctness." It will be observed that it is entirely independent of any trial.

It is evident that the approximations will be most rapid when $a^3 > b^2$; a condition which the irreducible case always involves.— Prof. E. W. Evans, Marietta College, Ohio.

5. *Problems in Imaginary Trigonometry.* — PROBLEM 1. To find an arc whose given sine is greater than radius.

Suppose the radius of the circle to be unity, and let $\sin\varphi = e$, and $e > 1$; we are to obtain a general expression for the value of φ.

For convenience, let $i = \sqrt{-1}$; and observe that

$$(1) \qquad i\cos\varphi = i\sqrt{1-\sin^2\varphi} = \sqrt{i^2+\sin^2\varphi}.$$

From the differentiation of circular functions, we have

$$d\sin\varphi = \cos\varphi\, d\varphi;$$

whence

$$(2) \qquad d\varphi = \frac{d\sin\varphi}{\cos\varphi} = \frac{i\,d\sin\varphi}{\sqrt{i^2+\sin^2\varphi}}.$$

Integrating, and then restoring the value of $\sqrt{i^2+\sin^2\varphi}$,

$$(3) \qquad \varphi = i\log[\sin\varphi + i\cos\varphi] + \text{constant}.$$

Now, if e be taken at its inferior limit, 1, φ will be 90°, and $i\log[\sin\varphi + i\cos\varphi]$ will be 0; whence

$$90° = \text{constant};$$

$$(4) \qquad \therefore\ \varphi = 90° + i\log[\sin\varphi + i\cos\varphi].$$

In order to eliminate $\sin\varphi$ and $\cos\varphi$ from this expression, let ψ be an arc, such that $\sin\varphi = \frac{1}{\cos\psi}$; then, from CHAUVENET'S Trig. Eq. 154, we have

$$(5) \quad \tan(45° + \tfrac{1}{2}\varphi) = \sqrt{\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)} = \sqrt{\left(\frac{\cos\psi+1}{\cos\psi-1}\right)} = -i\cot\tfrac{1}{2}\psi;$$

and

$$(6) \quad \tan(45° - \tfrac{1}{2}\varphi) = \sqrt{\left(\frac{1-\sin\varphi}{1+\sin\varphi}\right)} = \sqrt{\left(\frac{\cos\psi-1}{\cos\psi+1}\right)} = i\tan\tfrac{1}{2}\psi.$$

Subtracting (6) from (5), we have (CHAUVENET'S Trig. Eq. 157.

$$2\tan\varphi = -i(\tan\tfrac{1}{2}\psi + \cot\tfrac{1}{2}\psi).$$

Substituting in the second member, (CHAUVENET's Trig. Eq. 146 and 147,) and reducing, we have

$$\tan \varphi = -\frac{i}{\sin \psi};$$

whence

(7) $$i \cos \varphi = \frac{i \sin \varphi}{\tan \varphi} = -\tan \psi;$$

and

(8) $$\sin \varphi + i \cos \varphi = \frac{1}{\cos \psi} - \tan \psi = \frac{1 - \sin \psi}{\cos \psi} = \tan (45° - \tfrac{1}{2} \psi).$$

Whence, by substitution in (4), we obtain

(A) $$\varphi = 90° + i \log \tan (45° - \tfrac{1}{2} \psi),$$

the general expression sought.

PROBLEM 2. — To find an arc whose cosine is greater than radius.

Let $\cos \varphi' = e = \frac{1}{\sin \psi'}$, e being greater than radius. Then, by the course pursued in the first problem, we shall obtain

(1) $$\varphi' = -i \log (\cos \varphi' + i \sin \varphi').$$

Eliminating $\cos \varphi'$ and $\sin \varphi'$, we obtain, after the necessary reductions,

(B) $$\varphi' = -i \log \tan \tfrac{1}{2} \psi'.$$

In order to obtain a construction of equations (A) and (B), disregarding the factor, i, substitute $\cos^{-1} \frac{1}{e} = \psi$, and $\sin^{-1} \frac{1}{e} = \psi'$, and remembering that

$$45° - \tfrac{1}{2} \cos^{-1} \frac{1}{e} = \tfrac{1}{2} \sin^{-1} \frac{1}{e},$$

we have

(A′) $$\varphi = 90° + \log \tan \left(\tfrac{1}{2} \sin^{-1} \frac{1}{e}\right),$$

(B′) $$\varphi' = -\log \tan \left(\tfrac{1}{2} \sin^{-1} \frac{1}{e}\right).$$

Putting $t = \varphi - 90°$, and $t' = \varphi'$, and introducing into the sec-

ond members the modulus of common logarithms, and the value of radius in seconds of arc, we have

$$(a) \qquad t = 474942''.27 \log \tan\left(\tfrac{1}{2}\sin^{-1}\frac{1}{e}\right),$$

$$(b) \qquad t' = -474942''.27 \log \tan\left(\tfrac{1}{2}\sin^{-1}\frac{1}{e}\right),$$

in which the logarithms are common.

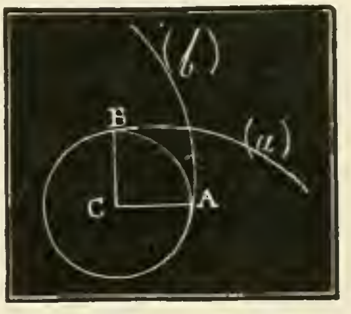
Now observe,

1st. For all values of e greater than 1, t is essentially negative and t' positive.

2d. If $e = 1$, $t = 0$, and $t' = 0$

3d. If $e = +\infty$, $t = -\infty^\circ$, and $t' = +\infty^\circ$.

Hence, each equation, (a) and (b), is the equation of a *spiral*, of which e is the radius vector, and t, or t', the measuring arc, whose origin is at a distance of unity from the origin of the spiral curve.



The positions of these spirals, relative to the circular arc which corresponds to values of e less than 1, when φ and φ' are real, are readily determined. Let the circular curve have its origin at A, and be reckoned in the direction of AB. Now, since $\varphi = 90°$, and $\varphi' = 0$, when $e = 1$, the origin of the spiral whose equation is (a) is at B, and the origin of the spiral whose equation is (b) is at A; and since t is essentially negative, and t' essentially positive, the directions of the curves will be *backward*, and *forward*, respectively, as shown in the figure.

To give a general idea of the relative increments of t, or t', and e, we add the following results, obtained by computation from (a) and (b).

If $\quad e = 1.005, \quad t = -(5° \, 44' \, 15''), \quad$ and $\quad t' = 5° \, 44' \, 15''$.

If $e = 1.5$, $t = -(55° \; 8' \; 32'')$, and $t' = 55° \; 8' \; 32''$.
If $e = 2.$, $t = -(75° \; 27' \; 21'')$, and $t' = 75° \; 27' \; 21''$.
If $e = 2.508642$, $t = 90°$, and $t' = 90°$.

— Prof. J. C. PORTER, Liberal Institute, Clinton, N. Y.

THE MOTIONS OF FLUIDS AND SOLIDS RELATIVE TO THE EARTH'S SURFACE.

[Concluded from Page 346.]

SECTION VII.

ON THE MOTIONS OF SOLID BODIES.*

82. It is proposed in this section to treat of the motions of projectiles and rotating bodies relative to the earth's surface, so far principally as they are influenced by the earth's rotation. By putting P and its derivatives in our fundamental equations equal 0, we get the equations of a projectile. Hence we obtain from equations (9), by restoring the value of α in § 2, the following general equations of the motions of a projectile relative to the earth's surface;—

$$\begin{aligned} D_t^2 r &= r(D_t\theta)^2 + r\sin^2\theta\,(2n + D_t\varphi)\,D_t\varphi - g, \\ (61)\qquad r D_t^2\theta &= -2\,D_t r\,.\,D_t\theta + r\sin\theta\cos\theta\,(2n + D_t\varphi)\,D_t\varphi, \\ r\sin\theta\,D_t^2\varphi &= -2\sin\theta\,(n + D_t\varphi)\,D_t r - 2r\cos\theta\,(n + D_t\varphi)\,D_t\theta. \end{aligned}$$

83. Let α, β, and γ be three rectangular co-ordinates of the projectile, of which α is vertical, β directed toward the south, and γ toward the east, and having their origin at the point of projection.

* Many of the results in this section have been given in a special paper on this part of the subject, published in the fifth volume of Gould's Astronomical Journal. For the sake of completeness, they are here deduced from more general equations and given again.

Let u, v, and w be the lineal velocities respectively in the directions of α, β, and γ, and n' the lineal velocity of the rotation of the earth's surface at the latitude of projection.

Also let R, θ', and φ' be the initial values, or the values at the point of projection, of r, θ, and φ, and u', v', and w' the initial values respectively of u, v, and w.

We shall then have, without any sensible error for the limited range of a projectile,

$$\alpha = r - R, \qquad \beta = R\,(\theta - \theta'), \qquad \gamma = R \sin\theta'\,(\varphi - \varphi'),$$

$$u = D_t r, \qquad v = R\, D_t\, \theta, \qquad w = R \sin\theta'\, D_t\, \varphi,$$

$$n' = R\, n \sin\theta' = \sin\theta' \times 1523.2 \text{ feet.}$$

By means of these equations, equations (61) are reduced to

$$D_t^2\, \alpha = \frac{v^2}{R} + \frac{(2\,n' + w)\,w}{R} - g,$$

$$(62) \qquad D_t^2\, \beta = -\frac{2\,u\,v}{R} + \frac{(2\,n' + w)}{R}\, w \cot\theta',$$

$$D_t^2\, \gamma = -\frac{2\,(n' + w)\,u}{R} - \frac{2\,(n' + w)\,v}{R} \cot\theta'.$$

84. In integrating these equations, g, and also v and w, may be regarded as constant, and equal to their initial values.

By a first integration we get, reckoning t from the time of projection,

$$D_t\, \alpha = u = u' - g t + \frac{v^2}{R}\, t + \frac{(2\,n' + w')\,w'}{R}\, t,$$

$$(63) \quad D_t\, \beta = v = v' - \frac{2\,u'\,v'}{R}\, t + \frac{(2\,n' + w')\,w' \cot\theta'}{R}\, t + \frac{2\,g\,v'}{R}\, t^2,$$

$$D_t\, \gamma = w = w' - \frac{2\,(n' + w')\,u'}{R}\, t - \frac{2\,(n' + w')\,v \cot\theta'}{R}\, t + \frac{2\,g\,(n' + w')}{R}\, t^2.$$

In integrating the last two equations, the value of $u = u' - gt$, neglecting the other two very small terms, was substituted before integration.

By a second integration we get

$$(64)\quad \begin{aligned} \alpha &= u' t - \tfrac{1}{2} g t^2 + \frac{v'^2}{2R} t^2 + \frac{(2n' + w')\, w'}{2R} t^2, \\ \beta &= v' t - \frac{u' v'}{R} t^2 + \frac{(2n' + w')\, w' \cot\theta'}{2R} t^2 + \frac{2 g v'}{3R} t^3, \\ \gamma &= w' t - \frac{(n' + w')\, u'}{R} t^2 - \frac{(n' + w')\, v' \cot\theta'}{R} t^2 + \frac{2 g (n' + w')}{3R} t^3. \end{aligned}$$

These are the complete equations of a projectile in terms of t and its initial velocity. The terms containing n' arise from the earth's rotation, and those having R in the denominator, from the earth's sphericity.

85. If we neglect the terms depending upon the earth's rotation and sphericity in the preceding equations, they may be reduced to

$$(65)\quad \begin{aligned} \alpha &= u' t - \tfrac{1}{2} g t^2, \\ s &= i t, \end{aligned}$$

in which s is the horizontal co-ordinate, having for its direction the initial direction of the projection, and i the horizontal velocity of projection. Hence the projectile in this case moves in a vertical plane. These are the equations of a projectile as given in elementary treatises, in which no account is taken of the earth's rotation, and in which the directions of gravity are supposed to be parallel. Eliminating t, we get the equation of a parabola, and hence, in this case, the motion is parabolic.

86. In the more general case, the terms containing n' and R in equations (64) deflect the motion from the parabola, and also from the vertical plane of projection. These terms, however, are very small, and their effects may be regarded merely as small perturbations from parabolic motion.

87. When the initial motion of projection is in a vertical direction, $v' = 0$, and $w' = 0$, and equations (64) become in this case

$$
(66) \qquad \begin{aligned}
\alpha &= u\,t - \tfrac{1}{2} g\,t^2, \\
\beta &= 0, \\
\gamma &= -\frac{n'\,u'}{R}\,t^2 + \frac{2\,g\,n'}{3\,R}\,t^3.
\end{aligned}
$$

Hence, since $\beta = 0$, an ascending or falling body projected vertically does not deviate to the north or south of a perpendicular; but the value of γ gives the deviation of such a body to the east or west, and depends entirely upon the earth's rotation.

If a body is projected vertically upward, when it arrives at its maximum height, $u = 0$, and the first of equations (63) then gives, neglecting the very small terms, $u' = g\,t$. Hence the last of the preceding equations gives

$$
(67) \qquad \gamma = -\frac{u'^3\,n'}{3\,R\,g^2} = -\frac{g\,n'}{3\,R}\,t^3
$$

for the deviation west when the body is at its greatest height.

If a body is let fall from a state of rest relative to the earth's surface, $u' = 0$, and we get in this case

$$
(68) \qquad \gamma = \frac{2\,g\,n'}{3\,R}\,t^3
$$

for the deviation of a falling body east of the perpendicular.

If a body is projected vertically upward, the last of equations (63) gives, when the body is at its maximum height, putting t' for the time of ascent,

$$
w = -\frac{2\,n'\,u'}{R}\,t' + \frac{2\,g\,n'}{R}\,t'^2.
$$

In applying the equation to the same body in falling, this value of w becomes the initial velocity, and must be substituted for w'. Hence we shall have

$$
D_t\gamma = -\frac{2\,n'\,u'}{R}\,t' + \frac{2\,g\,n'}{R}\,t'^2 - \frac{2\,n'\,u'}{R}\,t + \frac{2\,g\,n'}{R}\,t^2.
$$

Integrating,

$$\gamma = -\frac{2\,n'u'}{R}\,t'\,t + \frac{2\,g\,n'}{R}\,t'^2\,t - \frac{n'u'}{R}\,t^2 + \frac{2\,g\,n'}{3\,R}\,t^3.$$

When the body returns to the level from which it is projected, $t = t'$, and $g\,t = u'$, by which the preceding equation is reduced to

$$\gamma = -\frac{u'^3\,n'}{3\,R\,g^2}.$$

It deviates to the west in ascending, we have seen, by the same amount; and hence, if a body is projected vertically upward and returns, it falls to the west of the point of projection a distance equal to twice the preceding value of γ.

All the preceding deviations are small, amounting to only a few inches in a range of several hundred feet. Since the value of n' contains $\sin\theta$ as a factor, they are greatest at the equator and decrease as the sine of the polar distance.

88. When the projectile has a considerable horizontal range, the amount of deflection arising from the earth's rotation is much greater. If the earth had no rotation, the projectile would move in the same vertical plane. The deflection, therefore, from this plane depends upon the terms in equations (64) containing n'. Hence, if $\delta\,\beta$ and $\delta\,\gamma$ denote the effect of these terms upon the horizontal co-ordinates, we shall have very nearly, when the projectile does not have a great range in altitude,

$$(69)\qquad \begin{aligned} \delta\,\beta &= \frac{n'\,w'\cot\theta'}{R}\,t^2 = n\,w'\cos\theta'\,.\,t^2, \\ \delta\,\gamma &= -\frac{n'\,v'\cot\theta'}{R}\,t^2 = -\,n\,v'\cos\theta\,.\,t^2. \end{aligned}$$

The effect of the other terms containing n', we have seen, is very small, unless the range in altitude is great.

Let s be a horizontal co-ordinate in the direction of the horizontal range, and p a co-ordinate at right angles on the right, and i the

horizontal velocity of projection. Then, by resolving the preceding co-ordinates and velocities into the directions of s and p, we obtain

$$p = n\,i \cos\theta' t^2. \tag{70}$$

If a cannon-ball were to fly 3 miles in 12 seconds, the preceding equation, using the value of n in § 18, would give, at the parallel of 45°, about 10 feet for the value of p, or the deviation to the right of the initial direction.

89. If a body be supposed to move on the surface of the earth without friction, it would be continually deflected into a curve by the force $D_p F$, equation (53). If ϱ be the radius of curvature, and m the angular velocity about the centre of curvature, the centrifugal force will be ϱm^2, which must be put equal $D_p F$. Hence we have from equation (53)

$$\varrho m^2 = 2\,n\,v \cos\theta.$$

Also, since v is the lineal velocity of the moving body,

$$\varrho m = v.$$

Hence we have

$$\text{(71)} \qquad \begin{aligned} \varrho &= \frac{v}{2\,n\cos\theta}, \\ m &= 2\,n\cos\theta. \end{aligned}$$

When the range of motion is so small that $\cos\theta$ may be regarded as constant, ϱ and m are constant, and hence the body then moves in the circumference of a circle with a uniform angular velocity. If we put τ for the time of a revolution, we shall have

$$\tau = \frac{2\pi}{m} = \frac{2\pi}{n\cos\theta} = \sec\theta \times 1 \text{ day}. \tag{72}$$

Hence the time of a revolution is independent of the initial velocity of the body.

90. The gradual gyration of a vibrating pendulum is caused by this same deflecting force, and hence the time of gyration is the same as that of τ in the preceding equation.

91. If a body is forced to move in a straight line, the last form of the value of $D_p F$, equation (53), serves to compare the lateral pressure of this body with gravity. If we put $v = 60$, which is a velocity of about 40 miles per hour, the equation gives $D_p F = \frac{1}{5188} g$, at the parallel of 45°. Hence if a railroad train moves in a straight line 40 miles per hour at the parallel of 45°, the lateral pressure on the rails is $\frac{1}{5188}$ of its weight, and this is precisely the same for all directions, and not for the direction of the meridian only, as has been generally supposed.

92. We shall now examine the effect of the earth's rotation upon a rotating body of revolution, so suspended as to be free to turn about any axis of revolution passing through its centre of gravity. The same deflecting forces arising from the earth's rotation which act upon a free body having a motion relative to the earth's surface, must also act upon the different parts of a rotating body; and unless the sum of the moments of these forces with regard to any axis is equal to 0, it must have a tendency to turn the body around that axis, and consequently change the direction of the axis of rotation, unless it coincides with the axis of the moments.

93. Let A, B, and C be the sums of the moments which tend to turn the rotating body about the axes which are respectively perpendicular to the plane of the meridian, the prime vertical, and the horizon.

Also let r', θ', and φ' be the values of r, θ, and φ belonging to the centre of gravity of the rotating body.

We shall then have

$$
\begin{aligned}
A &= \int_m r\,(r - r')\, D_t^2\, \theta - \int_m r\,(\theta - \theta')\, D_t^2\, r, \\
(73) \qquad B &= \int_m r \sin\theta\,(r - r')\, D_t^2\, \varphi - \int_m r \sin\theta\,(\varphi - \varphi')\, D_t^2\, r, \\
C &= \int_m r^2 \sin\theta\,(\varphi - \varphi')\, D_t^2\, \theta - \int_m r^2 \sin\theta\,(\theta - \theta')\, D_t^2\, \varphi.
\end{aligned}
$$

Let ϱ be the distance of any particle of the rotating body from the axis of rotation; α, the distance from the centre of gravity of the plane passing through the particle perpendicular to the axis of rotation; φ, the angle between the plane of the meridian and a vertical plane passing through the axis of the rotating body; ψ, the angle of elevation of the axis above the horizon; i, the angular velocity of rotation; $it + \mu$, the angle between any particle and the vertical plane passing through the axis of rotation. We shall then have

$$r - r' = - \alpha \sin\psi + \varrho \cos\psi \cos(it + \mu),$$

$$(74) \quad r(\theta + \theta') = \alpha \cos\psi \cos\varphi + \varrho \sin\varphi \sin(it + \mu) + \varrho \sin\psi \cos\varphi \cos(it + \mu),$$

$$r \sin\theta (\varphi + \varphi') = - \alpha \cos\psi \sin\varphi + \varrho \cos\varphi \sin(it + \mu) - \varrho \sin\psi \sin\varphi \cos(it + \mu).$$

In taking the derivatives of r, θ, and φ with regard to t, since φ and ψ change very slowly in comparison with $(it + \mu)$, $D_t \varphi$ and $D_t \psi$ may be neglected in comparison with $D_t(it + \mu) = i$. Also r in the last two equations may be regarded as constant. Hence we get

$$D_t r = - i \varrho \cos\psi \sin(it + \mu),$$

$$r D_t \theta = i \varrho \sin\varphi \cos(it + \mu) - i \varrho \sin\psi \cos\varphi \sin(it + \mu),$$

$$r \sin\theta \, D_t \varphi = i \varrho \cos\varphi \cos(it + \mu) + i \varrho \sin\psi \sin\varphi \sin(it + \mu).$$

By substituting these values in the right-hand members of equations (61), and then substituting the values of the first members thus obtained, and also the values of the first members of equations (74), in equations (73), and integrating, we get

$$A = - n i (\sin\theta \sin\psi - \cos\theta \cos\psi) \int_m \varrho^2,$$

$$(75) \quad B = - n i \cos\theta \cos\psi \sin\varphi \int_m \varrho^2,$$

$$C = - n i \sin\theta \cos\psi \sin\varphi \int_m \varrho^2.$$

94. If in these equations we put $\varphi = 0$, and $\psi =$ the complement of θ, which is making the axis of the rotating body parallel with the earth's axis, we get $A = 0$, $B = 0$, and $C = 0$; and hence the axis of the rotating body in this position has no tendency to change its direction. In all other positions the small deflecting forces tend to change its direction.

95. If the rotating body is free to turn about any axis, the axis of rotation moves at right angles to the resultant of the forces which tend to change its direction, upon the same principle that the axis of an ordinary gyroscope gyrates horizontally at right angles to the direction of gravity. Hence it must gyrate around the position which it would have, if parallel with the earth's axis. If the body is free to turn about the axis only perpendicular to the meridian, its motion is oscillatory and determined by the first of equations (75). If it is free to turn about the axis only perpendicular to the prime vertical, its motions are determined by the second of equations (74); and if free to turn about the axis only perpendicular to the horizon, by the last of those equations; in both of which cases the motion is also oscillatory. These forces, however, are so small, even with the most rapid velocity of gyration, and in any experiments with the most delicate apparatus the resistances in comparison are so great, that the preceding motions are not observed, but only a tendency of the axis of the rotating body to assume the position in which all the forces are in equilibrium, which in the general case in which the axis is free to move in any direction, is a position parallel with the axis of the earth.

96. These deductions from theory are in exact accordance with some very delicate experiments made by FOUCAULT with a peculiar form of gyroscope, an account of which is given in the *American Journal of Science and Arts,* Second Series, Vol. XV. p. 263. See also Vol. XIX. p. 141.

SECTION VIII.

CONCLUSION.

97. In the first two sections of the preceding pages we endeavored to determine analytically the motions relative to the earth's surface which would satisfy all the conditions of a fluid surrounding the earth with a rotation on its axis, and also the figure which such a fluid must assume, on the hypothesis that the statical equilibrium of the fluid were very slightly disturbed by a difference of density between the equator and the poles. In the third section we proceeded in a similar manner with regard to a small circular portion of such a fluid, supposing that its statical equilibrium is disturbed by a difference of density between the centre and the external part. All the results obtained are on the hypothesis that the motions of the fluid are not resisted by the earth's surface, or, in the case of a circular portion of the fluid, by the resistance of the surrounding part. In the next three sections the results thus obtained were used to explain the general motions of the atmosphere and the ocean, the variation of barometric pressure at the earth's surface in different latitudes, the motions of cyclones, and the oscillations of the barometer, allowance being made for the modifying effects of friction or resistance from the earth's surface. In the last section the equations for the motions of solids were deduced from the general equations for the motions of fluids. and applied to the determination of the effect of the earth's rotation on the motions of a projectile and of a rotating body at the earth's surface.

98. For the sake of those who are not disposed to enter into a thorough analytical investigation of the subject, we propose to give in conclusion, without regard to quantitative results, a brief ex-

planation of the principal results at which we have arrived analytically, based upon well-known principles, and also to give a few additional items upon several parts of the subject.

99. First, with regard to the general motions of the atmosphere and the ocean, it is well understood that if one part of a fluid has a greater density than another, the difference of pressure causes a current at the bottom from the denser to the rarer part, and a counter-current at the top. Hence, if a fluid surrounding the earth is less dense in the equatorial than the polar regions, there must be a current at the earth's surface toward the equator, and a current above toward the poles, unless these motions are modified by other forces. If the earth had no rotation on its axis, this interchanging motion would be always in the direction of the meridians, and consequently there would be no motion of the fluid relative to the earth's surface, east or west. But the earth having a rotation on its axis, by the well-known principle of the preservation of areas, the part of the fluid moving toward the poles, since it becomes nearer the axis of rotation, must acquire a greater angular velocity of rotation than the earth, that is, it must acquire an eastward motion relative to the earth's surface. For the same reason the part moving toward the equator, since it becomes farther from the earth's axis, must acquire a westward motion relative to the earth. Hence it is evident that the atmosphere and the ocean, being less dense at the equator than at the poles, on account of a difference of temperature, must have an eastward motion toward the poles, and a westward motion in the equatorial regions. The velocity of this motion, on the hypothesis that it is not resisted by the earth's surface, and that its initial state was that of rest relative to the earth, is determined by equation (31), and the modifying effect of the resistance of the earth's surface has been treated in the fourth section.

100. If every part of a homogeneous fluid surrounding the earth

had the same angular motion of rotation with the earth, it would be in a state of statical equilibrium relative to the earth, having the same elliptical figure or outline, and its pressure upon the earth's surface would be everywhere very nearly the same. The part of the centrifugal force arising from the earth's rotation resolved in the direction of the meridians, which would be necessary to keep it in this state, is evidently $r n^2 \sin\theta \cos\theta$. If now any part of the fluid had a greater angular velocity of rotation expressed by $n + D_t\varphi$, this force would become $r(n + D_t\varphi)^2 \sin\theta \cos\theta$, and the fluid would press toward the equator with a force represented by the difference of these two expressions, which is $r(2n + D_t\varphi) \sin\theta \cos\theta\, D_t\varphi$. If $D_t\varphi$ were negative, unless it were greater than $2n$, this expression would be negative, and the pressure would be toward the pole. Now since the motion of the atmosphere toward the poles is eastward, and near the equator toward the west, it is evident that this pressure is exerted both from the poles and the equator toward the parallel where the atmosphere has no motion east or west, and consequently must cause an accumulation of the atmosphere there and a greater barometric pressure. The figure which the atmosphere must assume in consequence of these pressures is represented by Fig. 5.

101. From what precedes, if a body moves on the surface of the earth in the direction of the meridians, it is deflected to the right in the northern hemisphere, and the same is true if it moves in the direction of the parallels of latitude. Now since in any direction in which a body moves its motion must be compounded of these two motions, the resultant of the two deflecting forces belonging to each of these components must cause the body to be deflected to the right, in whatever direction it may move. In the southern hemisphere the deflection must be to the left. This result has been obtained analytically in § 32, and the amount of this deflecting force is given in equation (53).

102. By the preceding principle we may also explain the motion of a cyclone from the equator toward the poles, independently of our analytic results in § 31. The motion of the equatorial side of a cyclone in either hemisphere is always toward the east, and hence the deflecting force causes a pressure toward the equator, but that of the polar side being always toward the west, the deflecting force causes a pressure toward the pole. Now these deflecting forces being as the sine of the latitude, as may be seen from § 100, the pressure on the polar side toward the pole is greater than that on the other side toward the equator, and hence the cyclone moves in the direction of the greatest pressure. It is not to be supposed, however, that there is an actual transfer of all the atmosphere of a cyclone from the equator to the polar regions. For the motions and pressure of the cyclone being greater on the polar side, where the deflecting forces which cause it are greatest, its action upon the atmosphere in advance of it is greater than on the equatorial side, where these forces are much less, and hence new portions of the atmosphere are being continually brought into action on the one side, while the resistance of the earth's surface, and the adjacent portions of atmosphere on the other side, are continually overcoming the comparatively weak forces there, and destroying the gyratory motion of the cyclone; so that the centre of the cyclone is being continually formed in advanced portions of the atmosphere. Since many cyclones are more than one thousand miles in diameter, the difference in the violence of its action on the two sides is very considerable.

103. This same principle may be used to explain some of the motions of a rotating body. Suppose such a body be placed with its axis of rotation in any direction parallel with the horizon. Then the motions of the upper and the lower parts being in contrary directions relative to the earth, and the deflecting force in both be-

ing either to the right or the left, according to the hemisphere in which it is, gives the axis of rotation a tendency to assume a perpendicular position. But there are other forces beside these horizontally deflecting forces, so that all the forces which tend to change the position of the axis would not be in equilibrium with the axis in that position. For the equatorial side then would have a motion coinciding in direction with the motion of the earth's rotation, while the other side would have a motion the contrary way, and consequently the centrifugal force arising from the motion of the earth's rotation, combined with that of the rotating body, would be greater on the equatorial than on the polar side, and give the axis of the rotating body a tendency to move in the plane of the meridian. It might be easily shown, when the axis has a position parallel with the axis of the earth, that the forces which tend to change its direction are then in equilibrium, and consequently the axis, if free to turn in any direction, does not change its position.

104. We shall now give a little additional matter upon several points which would more properly have come in the preceding sections. It has been seen (§ 100), that, in consequence of the earth's rotation, the interchanging motion of the atmosphere between the equator and the poles gives rise to a force, represented by $r(2n + D_t \varphi) \sin\theta \cos\theta \, D_t \varphi$, by which this motion itself is counteracted. For instance, the motion toward the poles in the upper regions causes an eastward motion which gives rise to a force toward the equator, and which, consequently, counteracts the motion toward the poles, and the motion toward the equator produces a westward motion which gives rise to a force acting in the direction of the poles, which counteracts the motion toward the equator. The motion of the atmosphere, therefore, between the equator and the poles, is not produced by the whole force arising from the difference of density between the equator and the poles, but by a small dif-

ference only between the two forces. The difference of the preceding forces has been represented by W (§ 44), and it was shown in § 49 that it must in general be very small in comparison with those forces themselves. Hence if the earth had no rotatory motion, the force which produces this motion would be very much greater, and there would be a sweeping hurricane from the pole to the equator.

105. The approximate east or west motion of the atmosphere at the earth's surface, and also at the height of three miles, was computed for several parallels of latitude, from equation (60), neglecting the term W, and given in the table in § 48. The computation is based upon small differences of barometric pressure, and also upon the hypothesis that the difference of temperature between the equator and the poles is 60°; and was designed principally to give a general idea of what the motions must be in the upper regions, where the error arising from neglecting W must be the least. The atmosphere is extremely mobile, and consequently is very much disturbed by a great many local and temporary causes, which our investigation does not take into account, so that it is only the average velocity of a very changeable motion which can be compared with our results. There is no determination of such an average for the upper regions of the atmosphere, and even at the earth's surface this average, from the nature of the observations, cannot be accurately determined. It is, however, known from observations of the motions of the clouds, and also of balloons, that the eastward motion is much greater above than at the earth's surface, in accordance with our computed results. In Professor Coffin's Treatise on the Winds (*Smithsonian Contributions*, Vol. VII. p. 184) is a table of resultants determined from observations on the velocity and direction of the wind, from which the average east or west velocity may be determined for comparison with our computed results for the earth's surface. The average of all the observations in England and Scot-

land gives an eastward velocity of about six miles per hour, and the average of all in the Middle and Western United States gives an eastward velocity only a little more than two miles per hour. The computed velocity, according to our table, for the average latitude of the former observations, is about nine miles per hour, and for that of the latter, about four miles. Hence the computed results are from two to three miles less than those determined by observation, which is evidently the effect of the term W in equation (60), which was neglected. For since the atmosphere has a motion toward the poles in the middle latitudes, W is there positive (§ 49), and hence the neglect of it diminishes the results. This difference between the observed and computed results furnishes the value of W for those latitudes at the surface of the earth, and since W must be greatest there, the omission of it cannot cause an error of the same amount in the computed results for the upper regions.

106. With regard to the gyratory motion of the oceans, it may be further added here, that such gyrations are clearly demonstrated by the positions of the isothermal lines, as has been shown by Professor Dana, in a paper read at the twelfth meeting of the American Association for the Advancement of Science (*Proceedings*, Vol. XII. p. 77). According to this paper, the isothermal line of 68° F., in winter, extends, in the North Atlantic, from 56° N. on the American side, to 12° N. on the African, and in the South Atlantic, from latitude 31° S. on the South American coast, to 7° S. on the African side. Similar evidences are given of gyratory motions, in a less degree, in both the North and South Pacific, and also in the Indian Ocean.

107. The cause of the maximum accumulation of the atmosphere near the parallel of 30°, and the explanation of the southwest winds in the middle latitudes, which are produced by it, were first given in my former essay, published in the year 1856. At the meeting of the British Association for the Advancement of Science, in the year

1857, Mr. Thomson read a short paper in which he explains this accumulation of atmosphere, and the consequent reversion of the lower strata of the atmosphere in the middle latitudes, as arising from the centrifugal force of the eastward motion of the atmosphere, and illustrates the effect of such a force by means of a gyrating vessel of water in which the surface water recedes from the centre, while at the bottom there is a flowing toward it. In this paper nothing is said of the influence of the earth's rotation, and if he means that the effect is produced simply by the centrifugal force arising from the eastward motion of the atmosphere relative to the earth's surface, independent of the earth's rotation, the force would not be great enough to produce any sensible effect. For by examining the expression of the force which produces this effect, in § 100, it is seen that it depends principally upon the earth's rotation, since $D_t \varphi$, the angular velocity of the atmosphere relative to the earth, is small in comparison with $2n$, which is double the velocity of the earth's rotation. That the motion of the lower strata only in the middle latitudes is reversed, by this cause, as Mr. Thomson states, and as is represented by Fig. 5 in this paper, and not the motion of the whole atmosphere in those latitudes, as has been supposed, and as was represented in my first essay, is undoubtedly true; but instead of only a thin stratum next the earth's surface, as Mr. Thomson thinks, observations on the motions of the clouds indicate that this stratum extends to a considerable height.

The reader, who has also read my former essay, will notice one or two other slight changes in the results, but all the main points, which were imperfectly set forth in that essay, are fully established by the present more thorough investigation of the subject.

Cambridge, Mass., February, 1860.

Editorial Items.

We have received the following solutions of the Prize Problems in the May number of the Monthly: —

Harriet S. Hazeltine, Worcester, Mass., Probs. I., II.

J. G. Weinberger, Pennsylvania State Normal School, Millersville, Prob. I.

George B. Hicks, Cleveland, Ohio, Probs. III., IV., V.

Asher B. Evans, Madison University, N. Y., Probs. I., II., III., IV., V.

David Trowbridge, Perry City, N. Y., Probs. I., II., IV., V.

D. G. Bingham, Ellicotteville, N. Y., Prob. II.

Gustavus Frankenstein, Springfield, Ohio, Prob. IV.

Mathematical Monthly Notices.

Translation of the Sûrya-Siddhânta, A Text-Book of Hindu Astronomy, with Notes and Tables, and an Appendix containing additional Notes and Tables, Calculations of Eclipses, a Stellar Map, and Indexes. From the Journal of the American Oriental Society, Vol. VI. New Haven, 1860. 8vo. pp. 364. Sold by the Society's Agent, John Wiley, 56 Walker Street, New York.

The astronomy of the Hindus, though discussed at length by Bentley, Delambre, Biot, and various others, has never before been presented to us in the only satisfactory form, by a translation of one of their own treatises. We have had no means of knowing the real extent and character of Hindu science, or of comparing it with that of other nations to learn its origin. The present work furnishes that information, and is a very important contribution to the history of the exact sciences, — by far the most important ever published in this country. The first rough draft of the translation, as we learn from the preface, was made in India by Rev. E. Burgess, formerly Missionary of the A. B. C. F. M., aided by a native astronomer. The whole collected material was however placed in the hands of the committee of publication of the Oriental Society for revision, expansion, and reduction to a suitable form. It is not difficult to see in every line the careful, conscientious scholarship of Professor Whitney, and the lovers of the exact sciences owe to him the greater debt, that he, a philologist, has made so rich a contribution to their favorite department of knowledge.

The Sûrya-Siddhânta is one of the earliest and most esteemed of the many text-books of Hindu Astronomy. We are by no means, however, compelled to believe in the Hindu date, which makes it a revelation from the *Sun* given about 2,163,101 B. C. There are, however, indications of its existence as early as the sixth century of our era, and the astronomical data correspond best with even an earlier epoch.

The treatise is poetical, and consists of very brief elliptical rules for calculating the places of the heavenly bodies, solar and lunar eclipses, and other celestial phenomena. These rules would be wholly unintelligible without a teacher or commentary, or some knowledge beforehand of what should be said. The whole system presents a strange mixture of very exact knowledge and the most arbitrary and fanciful assumptions. The planetary theory is essentially that of the Greeks, given by Ptolemy in the Syntaxis, differing from it only in some minor particulars. In each the system of epicycles is fundamental. A full and careful comparison with the Syntaxis

is presented in the notes. It is only by such means that we can decide how far the Hindus were indebted to others for their astronomy. To show the accuracy of some of their determinations we quote (from p. 24) the following table of the times of sidereal revolutions, as given by different authorities.

Planet.	Sûrya-Siddhânta.	Siddhânta-Çiromaṇi.	Ptolemy.	Moderns.
	d. h. m. s.	d. h. m. s.	d. h. m. s.	d. h. m. s.
Sun, ...	365 6 12 36.56	365 6 12 9	365 6 9 48.6	365 6 9 10.75
Mercury, . .	87 23 16 22.26	87 23 16 41.52	87 23 16 42.9	87 23 15 43.89
Venus, . . .	224 16 45 56.19	224 16 45 1.93	224 16 51 56.8	224 16 49 7.99
Mars,. . . .	686 23 56 23.48	686 23 57 1.50	686 23 31 56.1	686 23 30 41.41
Jupiter, . . .	4,332 7 41 44.33	4,332 5 45 43.69	4,332 18 9 10.5	4,332 14 2 8.55
Saturn, . . .	10,765 18 33 13.25	10,765 19 33 56.48	10,758 17 48 14.9	10,759 5 16 32.22
Moon :				
Sid. revolut'n,	27 7 43 12.63	27 7 43 12.12	27 7 43 12.1	27 7 43 11.42
Synod. rev., .	29 12 44 2.80	29 12 44 2.28	29 12 44 3.3	29 12 44 2.89
Rev. of apsis,	3,232 2 14 53.43	3,232 17 37 5.98	3,232 9 52 13.6	3,232 13 48 29.64
" " node,	6,794 9 35 45.40	6,792 6 5 41.88	6,799 23 18 39.4	6,798 6 41 45.60

As a contrast to this exact knowledge, it may be mentioned that the positions of twenty-eight fixed stars are given, but eight of them have errors of over three degrees either in latitude or in longitude. We mention again, as characteristic of the system, that the nodes and apsides of all the planets are made to revolve, but with so slow a motion withal that even with the best modern instruments, it could not in some cases be detected in hundreds of years. We commend the note on pp. 26 – 28 to the attention of the curious.

This mixture of exact knowledge, gross error, and arbitrary assumption suggests the conclusion, which the arguments of Prof. Whitney prove, that the science of the Hindus was derived from the Greeks, being worked over by the borrowers to suit their fanciful mythology and chronology, and to serve as a book of rules for the astrologers and for the calculation of eclipses.

The origin of the Hindu division of the zodiac into twenty-seven or twenty-eight parts, and its connection with the Chinese zodiac and the Arabic lunar mansions is very fully discussed, and illustrated by a stellar map, and much light is thrown on this obscure subject.

The amount of the knowledge of the Hindus in arithmetic, geometry, and trigonometry are indicated by this treatise. The decimal notation and the sine of an arc were known. Angles are not mentioned in the treatise. A love of calculation is quite prominent. Multiplications and divisions, where the numbers have ten or twelve figures each, are quite frequent. Almost the only geometrical knowledge involved in the rules is that of the forty-seventh problem of Euclid and that of the proportionality of the sides of similar right-angled triangles. Of these, however, they make constant and dexterous use, solving, in fact, all ordinary cases of plane and spherical trigonometry.

The notes are quite full, being designed to satisfy the wants of two classes, the philologists who are not astronomers, and astronomers who are not philologists. We heartily commend this work to the attention of those who are interested in the history of astronomy or mathematics.

There are a great many other treatises of Hindu astronomy, and some of them are of considerable age. A comparison of these treatises would doubtless serve to show what points are peculiar to the Hindu astronomy, and what were borrowed from other systems. We sincerely hope that Prof. Whitney, who has shown himself so admirably qualified for the task, will place science under renewed obligations by undertaking it.

THE MATHEMATICAL MONTHLY.

Vol. II. . . . SEPTEMBER, 1860. . . . No. XII.

PRIZE PROBLEMS FOR STUDENTS.

I. If AB, CD be chords of a circle at right angles to each other, prove that the sum of the arcs AC, BD is equal to half the circumference.

II. Prove that the cube of any number and the number itself, being divided by 6, leave the same remainder.

III. It is required to divide a given right cone into two parts, having the ratio of m to n, by a plane making an angle of p degrees with the axis. — Communicated by Prof. Wray Beattie, Iowa Wesleyan University.

IV. In any spherical triangle,

$$\frac{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}{\cos S \cos (S-A) \cos (S-B) \cos (S-C)} = \frac{\cos a \cos b \cos c - 1}{\cos a \cos b \cos c + 1},$$

in which $s = \frac{1}{2}(a+b+c)$, $S = \frac{1}{2}(A+B+C)$. — Communicated by Prof. H. H. White, Harrodsburg, Ky.

V. Describe an ellipse having its foci at any points on one of the asymptotes of an equilateral hyperbola. The ellipse will cut each branch of the hyperbola and its conjugate in two real or imaginary points. Prove that a chord joining one pair of these points will be perpendicular to one of the other similar chords. Prove it first for a circle and then for an ellipse.

Solutions of these problems must be received by November 1, 1860.

NOTES AND QUERIES.

1. *Commensurable Sides of Right-angled Triangles nearly Isosceles.* — If in the equation,

$$(2\,a_n\,b_n)^2 + (a_n^{\,2} - b_n^{\,2})^2 = (a_n^{\,2} + b_n^{\,2})^2,$$

we put $a_1 = 1,\ b_1 = 0,$ then $0^2 + 1^2 = 1^2;$
$a_2 = 2,\ b_2 = 1,$ " $4^2 + 3^2 = 5^2;$
$a_3 = 5,\ b_3 = 2,$ " $20^2 + 21^2 = 29^2;$
$a_4 = 12,\ b_4 = 5,$ " $120^2 + 121^2 = 169^2;$

and in general, if $a_n = 2\,a_{n-1} + b_{n-1},\ b_n = a_{n-1}$, then,

$$(4\,a_{n-1}^2 + 2\,a_{n-1}\,b_{n-1})^2 + (3\,a_{n-1}^2 + 4\,a_{n-1}\,b_{n-1} + b_{n-1}^2)^2 = (5\,a_{n-1}^2 + 4\,a_{n-1}\,b_{n-1} + b_{n-1}^2)^2.$$

Also,

$$(2\,a_n\,b_n) - (a_n^2 - b_n^2) = (a_{n-1}^2 - b_{n-1}^2) - (2\,a_{n-1}\,b_{n-1})\,.$$
$$= (2\,a_{n-2}\,b_{n-2}) - (a_{n-2}^2 - b_{n-2}^2) = \ldots\ldots = \pm 1.$$

Additional values of $a_n,\ b_n$ give

$$696^2 + 697^2 = 985^2$$
$$4060^2 + 4059^2 = 5741^2$$
$$23660^2 + 23661^2 = 33461^2$$
$$159140520^2 + 159140519^2 = 225058681^2,$$

&c. &c. &c.

— Lucius Brown, Fall River, Mass.

2. *Note on the Equation*

$$(a^2 + b^2)^2 = (a^2 - b^2)^2 + (2\,a\,b)^2.$$

The general problem is to find two square numbers whose sum or difference shall also be a square number; and it may be solved by the Diophantine Analysis.

Let x be the greater, and y the lesser number; then,

(1) $$x^2 - y^2 = \square = (x - ny)^2 = x^2 - 2\,n\,x\,y + y^2,$$

in which n is any arbitrary entire or fractional number.

From (1) we have

$$x = \frac{n^2 + 1}{2\,n}\,y.$$

We may now put y equal n, $2\,n$, or $3\,n$, &c.; but to make the value of x as simple as possible, let $y = 2\,n$; then $x = n^2 + 1$ and $\sqrt{(x^2 - y^2)} = n^2 - 1$.

Now, by substituting any numbers for n in these equations, we shall obtain three numbers, satisfying the conditions of the problem. Suppose $n = 4$; then $17^2 = 8^2 + 15^2$; and if in the equation

$$(n^2 + 1)^2 = (n^2 - 1)^2 + 4\,n^2$$

we put $n = \frac{a}{b}$, it becomes

$$\left(\frac{a^2 + b^2}{b^2}\right)^2 = \left(\frac{a^2 - b^2}{b^2}\right)^2 + \frac{4\,a^2}{b^2},$$

or,

$$(a^2 + b^2)^2 = (a^2 - b^2)^2 + 4\,a^2\,b^2.$$

Hence, by assuming any two numbers for a and b, we shall find that $a^2 + b^2$ is the hypothenuse, and $a^2 - b^2$, $2\,a\,b$, the sides of a right-angled triangle. — Prof. D. Wood, University of Michigan, Ann Arbor.

3. *On the Logarithmic Solution of Cubic Equations.* — In the December number of the Monthly, as in various other publications, attention has been called to the trigonometric or logarithmic solution of equations. The subject appears of sufficient interest to merit further attention; and it is here proposed to combine the methods in a more systematic form, with such additional precepts as will adapt the operation even to persons who have attended only to the first principles of the logarithmic Tables. It is plain that all equations of the third degree, or cubics, may be distributed into three general classes, represented as follows: —

First class, ... $x^3 + a\,x = b.$
Second class, ... $x^3 + a'\,x^2 = b'.$
Third class. . . . . . $x^3 + a\,x^2 + a_1\,x = b.$

To change an equation of the second class into the form of the first, let $x = \frac{1}{y}$. Substituting this value for x, and reducing,

$$y^3 - \frac{a'}{b'}\,y = \frac{1}{b'}.$$

The value of y can here be found, as in the first class, the reciprocal of which will be the value of x.

Again, by adding and subtracting equal quantities, the equation of the third class can take the form,

$$\left(x + \frac{a}{3}\right)^3 + \left(a_1 - \frac{a^2}{3}\right)\left(x + \frac{a}{3}\right) = b + \frac{a\,a_1}{3} - \frac{2\,a^3}{27}.$$

Making $x + \frac{a}{3} = z$, we may find the value of z, as in the first class, and then determine x from the relation

$$x = z - \frac{a}{3}.$$

Referring now to the investigation in the Mathematical Monthly, Vol. II., p. 85 – 88, let l denote the common logarithm of the quantity before which it is placed; and the following rule is obtained.

For the first class of cubics having the form,

$$x^3 + a\,x = b,$$

where a, b denote either positive or negative numbers.

Regarding a, b as positive numbers, find the quantity

$$L = 1\tfrac{1}{2}\,.\,l\,\frac{a}{3} + 10 - l\,\frac{b}{2}.$$

I. *When* a *in the given equation is positive;* find

$$l\tan v = L, \qquad\qquad l\tan u = \tfrac{1}{3}\left(20 + l\tan\frac{v}{2}\right),$$

$$l\,x = l\,2 + \tfrac{1}{2}\,l\,\frac{a}{3} + l\cot 2\,u - 10.$$

II. *When* a *is negative,* and the index of L less than 10;

$$l \sin v = L, \qquad l \tan u = \tfrac{1}{3}\left(20 + l \tan \tfrac{v}{2}\right),$$

$$l\,x = l\,2 + \tfrac{1}{2} \cdot l\,\tfrac{a}{3} + 10 - l \sin 2\,u.$$

III. *When* a *is negative,* and the index of L is 10, or more ;

$$l \cos v = 20 - L, \qquad l\,x = l\,2 + \tfrac{1}{2} \cdot l\,\tfrac{a}{3} + l \cos \tfrac{v}{3} - 10.$$

Here it should be carefully noted to *apply to the values of* x, *thus far, the same algebraic sign with that of* b, *on the right-hand side of the given cubic.* But the two remaining values of x should be taken with a sign contrary to that of b :

$$l\,x = l\,2 + \tfrac{1}{2}\,l\,\tfrac{a}{3} + l \cos\left(60^\circ - \tfrac{v}{3}\right) - 10,$$

and

$$l\,x = l\,2 + \tfrac{1}{2}\,l\,\tfrac{a}{3} + l \cos\left(60^\circ + \tfrac{v}{3}\right) - 10.$$

With the foregoing precepts, the auxiliaries v, u must always be taken positively, and not exceeding 90°.

Example.

Find the value of x, when $x^3 + 15\,x = 4$.

[To find L and u.]

$l\left(\frac{a}{3} = 5\right) =$	0.698970
$\frac{1}{2}\,l\,\frac{a}{3} =$	0.349485 . . . .
	+10.
$-\,l\left(\frac{b}{2} = 2\right) =$	−(0.301030)
$l \tan v$, 79° 51′ 28″....	10.747425 $= L$.
$l \tan \frac{v}{2}$, 39° 55′ 44″, 3	9.922720
	20.
$l \tan u$, 43° 18′ 6″.5....	9.974240
$2\,u =$ 86° 36′ 13″.	

[To find x.]

$l\,2 =$	0.301030
	0.349485
$l \cot 2\,u =$	8.773405
	−10.
l .265412	1.423920

$x = +0.265412.$

In this example, a or 15 is positive, showing that the constant L is to be considered a logarithmic tangent, and that the formulas of I.

are to be employed; this has been strictly done in the above operation, as will be seen by comparison. And since b or 4 is positive, the value of x in this case is also positive.

Further examples might easily be multiplied, were it necessary; but the preceding rule, within a short compass, appears sufficiently plain and comprehensive. — L. W. Meech, Assistant, U. S. Coast Survey, Preston, Ct.

ON THE INDETERMINATE ANALYSIS.

By Rev. A. D. Wheeler, Brunswick, Maine.

[Concluded from Page 195.]

Proposition XV. The whole number of possible equations in such a series, or for every integral value of $c < ab + 1$ is expressed by the formula $\frac{ab - (a + b - 1)}{2}$.

Demonstration. Dividing ab by a, we have $b =$ the whole number of impossible equations in which c is a multiple of a; and dividing by b, we have $a =$ the whole number of impossible equations in which it is a multiple of b. (Prop. IX., Cases 3 and 4.) But it will be seen that the impossible equation $ax + by = ab$ has thus been reckoned twice, when it should have been reckoned but once. Deducting 1 on account of this repetition, we shall have $a + b - 1$ for the whole number of impossible equations in which c is a multiple of either a or b. If this number be subtracted from ab, there will remain $ab - (a + b - 1)$, which, according to Prop. XIV., includes an equal number of possible and of impossible equations. Therefore $\frac{ab - (a + b - 1)}{2}$ expresses all the possible equations in which $c < ab + 1$.

Prop. XVI. The whole number of equations in such a series, which will admit of one, and only one solution, is denoted by the product ab.

Dem. All values of $c > 2ab$ will admit of two solutions, according to Prop. VIII. It may be shown, as in the preceding proposition, that all the values of $c < 2ab + 1$, which will admit of two solutions, may be expressed by the same formula, $\frac{ab - (a+b-1)}{2}$; and this, subtracted from ab, representing the whole interval between ab and $2ab$, gives $ab - \frac{ab - (a+b-1)}{2} = \frac{ab + a + b - 1}{2}$ for the whole number of equations within that interval, admitting of but one solution. Adding the expressions for the number of solutions admitting one solution only above ab and below it, we have

$$\frac{ab + (a+b-1)}{2} + \frac{ab - (a+b-1)}{2} = ab;$$

which was to be shown.

Prop. XVII. Let there be any numbers of independent equations, having the same coefficients to x and y. The sum of all the solutions contained in the whole number of equations may be found by dividing the sum of the values of c by ab, and adding the number of solutions contained in the several remainders, according to the rule which is given under Prop. XIII.

Adding the equations

$$ax + by = nab + r, \qquad \text{(A)}$$
$$ax' + by' = n'ab + r', \qquad \text{(B)}$$
$$ax'' + by'' = (n'' - 1)\,ab + r'' \text{ \&c.} \qquad \text{(C)}$$

we obtain

$$ax_1 + by_1 = ab\,(n + n' + n'' - 1) + r + r' + r'' \text{ \&c.} \qquad \text{(D)}$$

Subtracting the remainders, we have the equation,

$$ax_{,,} + by_{,,} = ab\,(n + n' + n'' - 1) = c;$$

and dividing c by ab, we obtain the quotient $n + n' + n'' - 1$, which gives the number of solutions in (D) with reference to the

remainders. But in equation (A) we have n solutions; in equation (B), n' solutions; and in equation (C), $n'' - 1$ solutions; and the sum of these $= n + n' + n'' - 1$, the same expression which we obtained from (D). Now if we ascertain the number of solutions in these several remainders, and add it to the number obtained from (D), we shall have all the possible solutions belonging to the several equations (A), (B), and (C).

Remark.— Several of the preceding propositions will be found very convenient in proposing questions which shall have any desired conditions, and several of them will very much facilitate the operations which are necessary to be performed in the solution of problems which are to follow. They may easily be verified by substituting numbers for letters. YOUNG'S Algebra contains more than one erroneous proposition relating to this subject, to which demonstrations are professedly given both by the author and the American editors.

PROBLEMS.

I. To find the successive remainders in any Arithmetical Progression, without continuing the operation of division.

1. Suppose the progression to be ascending, and call the first remainder r.

The general expression for any term in this progression is $a + (n - 1)\, d$; and that for the next succeeding term will be $a + (n - 1)\, d + d$. Let D be the divisor, and let

$$\frac{a + (n - 1)\, d}{D} = Q + \frac{r}{D}.$$

Then will $a + (n - 1)\, d = D\, Q + r$. By substitution, we have

$$\frac{a + (n - 1)\, d + d}{D} = \frac{D\, Q + r + d}{D} = Q + \frac{r + d}{D}.$$

That is to find the next succeeding remainder, add the difference of

the progression to the remainder already found, rejecting the divisor as often as necessary.

2. Suppose the progression to be descending. Then by a process entirely similar, the remainder of the next succeeding term will be $r - d$. That is, from the remainder already found, the difference of the progression must be subtracted, and the divisor added as often as necessary. The reason for rejecting or adding the divisor may be readily shown. For $\frac{r + d - D}{D}$ still gives the same remainder $r + d$, and $\frac{D + r - d}{D}$ the same remainder $r - d$.

Illustration. — Let $D = 11$, $d = 7$, $a = 16$, $n = 8$. Then we have the progression

$$16, \quad 23, \quad 30, \quad 37, \quad 44, \quad 51, \quad 58, \quad 65\,;$$

and the remainders will be

$$5, \quad 1, \quad 8, \quad 4, \quad 0, \quad 7, \quad 3, \quad 10.$$

These are readily obtained by the rule, as follows: $16 - 11 = 5$; $5 + 7 - 11 = 1$; $1 + 7 = 8$; $8 + 7 - 11 = 4$; $4 + 7 - 11 = 0$; $0 + 7 = 7$; $7 + 7 - 11 = 3$; $3 + 7 = 10$.

Reversing the progression, we obtain the same remainders by reversing the process.

This method will serve to facilitate the operations, especially when the terms of the progression are large.

If we wish to find any particular term of these remainders, without finding all that precede, we have merely to make use of the formula

$$\frac{a + (n - 1)\,d}{D} = Q + \frac{r}{D},$$

and r will be found at once. For example,

Let $$n = 6.$$

Then we have $$\frac{5 + 5.7}{11} = 3 + \frac{7}{11}.$$

Therefore we have for the 6th term, $r = 7$, which is the same as that given above.

II. To solve the equation $ax \pm by = c$.

FIRST METHOD.

When a or b is a divisor of c, and one is prime to the other.

As a will divide c, it must divide its equal, $ax + by$; and as it divides ax, it must also divide by (Prop. I.). But it cannot divide b, since a and b are prime to each other. Therefore it must divide y (Prop. III.), and the least value of y will be a; the next, $a + a$, or $2a$; the next, $3a$, &c.

Hence one of the values of x or y may be determined by simple inspection, and the corresponding value of the other unknown may be readily deduced from the equation. Then by the addition or subtraction of a or b, as the case requires, all the other values may be obtained.

When a or b is not a divisor of c, but has a factor in common with it, then it may be shown in the same way that y is equal to that factor or to some multiple of it.

In the first case, however, we must have $c > ab$ (Prop. IX., Case 3), or the solution in whole numbers will not be possible.

Examples.

Let $5x + 7y = 40$, then $y = 5$ and $x = 1$.

Let $6x + 7y = 39$, then $y = 3$, which is a factor of 6 and of 39.

SECOND METHOD.

When a, b, and c are all prime to each other, and the numbers are not large.

Let $ax + by = c$. Then $x = \frac{c - by}{a}$ = an integer. Divide c

by a and note the remainder. Find the successive remainders resulting from the division of by by a, as in Problem I. The term which gives the same remainder as c is the value of y.

As $\frac{c - by}{a}$ is the same with $-\frac{by - c}{a}$, the signs of this expression may be changed when more convenient.

Again, let $ax - by = c$. Then will $x = \frac{c + by}{a}$; and the value of y may be found in the same manner, except that it will be necessary to find a remainder $= a - r$, in order to cancel the remainder resulting from c.

Examples.

1. Let $11x + 13y = 141$. Then

$$x = \frac{141 - 13y}{11} = 12 - y + \frac{9 - 2y}{11}.$$

Substituting for y the successive values 1, 2, 3, &c., we find that $y = 10$ will give the remainder 9. Hence 10 is its true value. The remainders, as found by Problem I., are 2, 4, 6, 8, 10, 1, 3, 5, 7, 9; and 9 is found in the 10th term of the series.

2. Let $17x - 23y = 7$. Then $x = \frac{7 + 23y}{17}$, and we require the remainder $17 - 7 = 10$. The series will be 6, 12, 17, 13, 2, 8, 14, 3, 9, 15, 4, 10. The remainder 10 is found in the 13th term. Therefore $y = 13$. This method is expeditious when the divisor is small.

THIRD METHOD.

1. To solve the indeterminate equation $ax - by = 1$.

By the theory of Continued Fractions, if any two consecutive approximations be reduced to a common denominator, the difference of their numerators will always be 1; and this difference will be positive, or negative, according as the number of quotients in the reduction is even or odd.

If a and b are prime to each other, and the fraction $\frac{a}{b}$ be reduced

to the form of a continued fraction, and the several approximate values obtained, the least value will be the fraction $\frac{a}{b}$ itself; and the next preceding may be represented by the ascertained values $\frac{y'}{x'}$. Now, in accordance with what has been stated,

$$\frac{a}{b} - \frac{y'}{x'} = \frac{a\,x' - b\,y'}{b\,x'} = \frac{\pm 1}{b\,x'};$$

or, rejecting the denominators, $a\,x' - b\,y' = \pm 1$, the upper sign denoting that the number of quotients in obtaining x' and y' is even, the lower that it is odd.

If then we make $x = x'$ and $y = y'$, we have in the first case $a\,x' - b\,y' = 1$; and the problem is solved. In the second case, we have $a\,x' - b\,y' = -1$. But by changing x' into $b - x'$, and y' into $a - y'$, we obtain

$$a\,x - b\,y = a\,(b - x') - b\,(a - y') = -a\,x' + b\,y' = +1;$$

and this case is also solved.

Hence the following rule: —

Find x' and y' by the process for finding the greatest common divisor, and the rule for Continued Fractions. Then, if the number of quotients be even, x' will be the value of x, and y' will be that of y; but if the number be odd, $b - x'$ and $a - y'$ will be their respective values.

Examples.

1. $12\,x - 17\,y = 1$.

$$\begin{array}{r|r|l} 12 & 17 & 1 \\ 10 & 12 & 2 \\ \hline 2 & 5 & 2 \\ & 4 & 5 = y' \\ \cline{2-2} & 1 & 7 = x' \end{array}$$

Here the number of quotients is odd. Therefore $x = 17 - 7 = 10$, and $y = 12 - 5 = 7$. Whence we have

$$a\,x - b\,y = 12\,.\,10 - 17\,.\,7 = 1,$$

$$2\,.\,2 + 1 = 5 = y', \qquad \text{and} \qquad 5\,.\,1 + 2 = 7 = x'.$$

2. $10x - 13y = 1$.

$$\begin{array}{r|r|l} 10 & 13 & 1 \\ 9 & 10 & 3 = y' \\ \hline 1 & 3 & 4 = x' \end{array}$$

Here the number of quotients is even. Hence, $x = x' = 3 \,.\, 1 + 1 = 4$, and $y = y' = 3$.

Rule. — Multiply the last quotient by that which immediately precedes, and add 1. Multiply this result by the last quotient not used, and add the preceding sum. Proceed in this manner till all the quotients have been employed. The last result will be x'; the one before it y', When there are but two quotients, there will be but one multiplier, and consequently the latter of the two will express the value of y'.

The whole operation is simple when once understood, and may be performed mentally, if the numbers are not too large.

2. To solve the equation $ax - by = c$.

Find x', and y', as before. Then will $ax' - by' = 1$, or $-ax + by = 1$, according as the number of quotients is even or odd.

1. Let $x = cx'$, and $y = cy'$. The equation becomes

$$ax - by = acx' - bcy' = c\,(ax' - by') = c\,;$$

and the conditions of it are satisfied.

2. Let $x = c\,(b - x')$, and $y = c\,(a - y')$. Then we have

$$ax - by = ca\,(b - x') - cb\,(a - y') = c(-ax' + by') = c\,;$$

and the conditions in this case are also satisfied.

3. To solve the equation $ax + by = nab + r$.

Find x', and y', as before; and suppose the number of quotients to be even. Let $x = rx'$, and $y = na - ry$. Then will

$$ax + by = arx' + nab - bry' = nab + r\,(ax' - by') = nab + r\,;$$

and this case is solved.

Suppose the number of quotients to be odd, and let $x = nb - rx'$, and $y = ry'$. Then

$$ax + by = nab - arx' + bry = nab + r(-ax' + by') = nab + r.$$

Wherefore this case also is solved.

The following formulæ are sufficient for the solution of all these cases : —

1. $ax + by = nab + r$; $x = rx'$ and $y = na - ry'$; when the quotients are *even.*
2. $ax + by = nab + r$; $x = nb - rx'$ and $y = ry'$; when the quotients are *odd.*
3. $ax - by = c$; $x = cx'$ and $y = cy'$; when the quotients are *even.*
4. $ax - by = c$; $x = c(b - x')$ and $y = c(a - y')$; when the quotients are *odd.*

Examples.

1. $8x + 13y = 319 = \overset{n}{3} \,.\, \overset{a}{8} \,.\, \overset{b}{13} - +\overset{r}{7}$

$$\begin{array}{l} 1 \\ 1 \\ 1 \\ \underline{1} \\ 2 \\ 3 = y' \\ 5 = x' \end{array}$$

$$x = rx' = 35.$$

$$y = na - ry' = 24 - 21 = 3.$$

$$8x + 13y = 8 \,.\, 35 + 13 \,.\, 3 = 319.$$

2. $10x + 17y = 689 = \overset{n}{4} \,.\, \overset{a}{10} \,.\, \overset{b}{17} + \overset{r}{9}.$

$$\begin{array}{l} 1 \\ 1 \\ \underline{2} \\ 3 = y' \\ 5 = x' \end{array}$$

$$x = nb - rx' = 68 - 45 = 23. \qquad y = ry' = 27.$$

$$ax + by = 10 \,.\, 23 + 17 \,.\, 27 = 689.$$

3. $\overset{a}{7}x - \overset{b}{10}y = \overset{c}{11}.$

$$\begin{array}{l} 1 \\ \underline{2} = y' \\ 3 = x' \end{array}$$

$$x = cx' = 33. \qquad y = cy' = 22.$$

$$ax - by = 7 \,.\, 33 - 10 \,.\, 22 = 11.$$

4. $7x - 12y = 5.$

$$\begin{array}{l} 1 \\ 1 \\ \underline{2} \\ 3 = y' \\ 5 = x' \end{array}$$

$$x = c(b - x') = 5(12 - 5) = 35.$$

$$y = c(a - y') = 5(7 - 3) = 20.$$

$$7x - 12y = 7 \,.\, 35 - 12 \,.\, 20 = 5.$$

4. Problem relating to three unknown quantities.

To determine the number of solutions which the equation

$$ax + by + cz = d,$$

will admit of in integers.

SOLUTION.

If we put for z the numbers 1, 2, 3, &c., in their natural order, we shall obtain the equations,

$$\begin{aligned} ax + by &= d - c, \\ ax' + by' &= d - 2c, \\ ax'' + by'' &= d - 3c, \text{ \&c.} \end{aligned}$$

until we obtain, at length,

$$ax_{,} + by_{,} = d - nc < a + b,$$

which will not admit of a solution.

Now the number of solutions which each of these equations can have, may be readily ascertained by the rule under Prop. XVI.; and as we give to z all its possible values, we shall evidently, by adding together the number belonging to each successive equation, obtain all the solutions required by the problem.

Example.

Let $3x + 5y + 7z = 100$. Giving to z the values of 1, 2, 3, &c., in succession, we have the equations,

$$\begin{aligned} 3x + 5y &= 93, \\ 3x' + 5y' &= 86, \\ 3x'' + 5y'' &= 79, \text{ \&c.} \end{aligned}$$

Now,

$\frac{d-c}{ab}$	$= 93 \div 15 = 6$,	with the	remainder	3,	and gives	6 solutions.
$\frac{d-2c}{ab}$	$= 86 \div 15 = 5$,	"	"	11,	"	6 "
$\frac{d-3c}{ab}$	$= 79 \div 15 = 5$,	"	"	4,	"	5 "
&c.	$72 \div 15 = 4$,	"	"	12,	"	4 "
	$65 \div 15 = 4$,	"	"	5,	"	4 "
	$58 \div 15 = 3$,	"	"	13,	"	4 "
	$51 \div 15 = 3$,	"	"	6,	"	3 "
	$44 \div 15 = 2$,	"	"	14,	"	3 "
	$37 \div 15 = 2$,	"	"	7,	"	2 "
	$30 \div 15 = 2$,	"	"	0,	"	1 "
	$23 \div 15 = 1$,	"	"	8,	"	2 "
	$16 \div 15 = 1$,	"	"	1,	"	1 "
	$9 \div 15 = 0$,	"	"	9,	"	0 "

Wherefore the whole number of solutions is . . . 41

When d is very large, and a, b, and c are very small, this method becomes laborious. It may, however, be greatly simplified.

1. The terms $d - c$, $d - 2c$, $d - 3c$, &c. are those of Arithmetical Progression; and as we have the first term, last term, and difference, we can easily find the sum of all the terms. Call this sum S.

2. As the remainders must necessarily recur in periods (Prop. V., R. 2), and as it is easy to find all the remainders in a period (Problem I.), we can readily obtain the sum of all the remainders, by multiplying the sum in one period by the number of periods and adding those in the partial period. Call this sum S'.

3. The number of solutions contained in the remainders may be found in the same way, by first finding those in one period, and then multiplying by the number of periods. Call it S''.

Then the whole number of solutions $= \frac{S - S'}{ab} + S''$.

When the period returns after ab terms, as it often does, the sum

of the remainders in a period will be equal to the sum of an arithmetical progression, whose first term is one, and difference one, and number of terms $a\,b$. It will therefore be expressed by the formula $\frac{ab\,(ab+1)}{2}$. Also the number of possible solutions in a period will be expressed by the formula $\frac{ab-(a+b-1)}{2}$ (Prop. XV.).

Thus, taking the last example,

$$
\begin{array}{llrlrl}
 & 3x+5y+7z= & 100 & & 15 & =ab \\
7)93 & 3x+5y= & 93 & =l & 1 & \\
\;\;13-2 & & 2 & =a & 2)\overline{16} & =ab+1 \\
n=14,\quad a=2 & & \overline{95} & =a+l & 8 & =\frac{ab+1}{2} \\
 & & 7 & =\frac{1}{2}n & 15 & \\
 & & \overline{665} & & \overline{120} & \frac{ab\,(ab+1)}{2} \\
 & & 110 & & 10 & \\
\frac{15-(3+5-1)}{2}=4 & & 15)\overline{555} & & \overline{110} & \\
 & & \overline{37} & & & \\
 & & 4 & & & \\
 & & \overline{41} & & &
\end{array}
$$

Example 2. To find the whole number of solutions that the equation $7x + 9y + 23z = 999$ will admit of.

Operation. — Calling $z = 1$, we have

$7x + 9y = 9976 = l,\qquad 9976 \div 23 = 433\frac{17}{23},\qquad \therefore\ n = 434;$

since the remainder 17 also constitutes a term, $\therefore a = 17$. But $(a+l)\frac{n}{2} = S = (17 + 9976)\,.\,217 = 2168481,\quad 434 \div 63 = 6\frac{56}{63}$; that is, 6 complete periods of remainders, and 56 remainders belonging to another period, or $63 - 56 = 7$ remainders that are wanting to complete another period.

$(63 + 1) \div 2 = 32.$

$32\,.\,63 = 2016 =$ the sum of the remainders in one period.

$2016\,.\,6 = 12096 =$ the sum in six periods.

$(9976 - 62 \,.\, 23) \div 63 = 135\frac{45}{63}$.

$\therefore$ 45 = the last remainder in a period, as 22 = the first.

Then, by Problem I., the 7 previous remainders will be

$$45 + 5 + 28 + 51 + 11 + 34 + 57 = 231.$$

Then $2016 - 231 = 1785$ = the sum of the 56 remainders in the partial period, and $12096 + 1785 = 13881 = S'$ = the sum of all the remainders. Then

$S - S' = 2168481 - 13881 = 2154600$. $(S - S') \div ab = 2154600 \div 63 = 34200$ = the number of solutions, exclusive of those comprehended in the remainders.

$\dfrac{ab - (a + b - 1)}{2} = (63 - (7 + 9 - 1)) \div 2 = 24$ = the number of solutions in the remainders of one period.

$24 \,.\, 6 = 144$ = the number in 6 periods.

$24 - 3$ (3 solutions are contained in the 7 remainders) $= 21$ = the number of solutions in the partial period.

Then

$144 + 21 = 165 = S''$; and $34200 + 165 = 34365$ = the answer.

5. To find the least whole number, which, being divided by given numbers, shall leave given remainders.

Suppose the number to be x, the divisors d, d', d'', &c., and the remainders r, r', r'', &c. Then each of the quantities, $\frac{x - r}{d}$, $\frac{x - r'}{d'}$, $\frac{x - r''}{d''}$, &c., is an integer. Reduce them to a common denominator, and they become

$$\frac{d'\,d''\,x - d'\,d''\,r}{d\,d'\,d''}, \qquad \frac{d\,d''\,x - d\,d''\,r}{d\,d'\,d''}, \qquad \frac{d\,d'\,x - d\,d'\,r}{d\,d'\,d''};$$

and having the same values, they still remain integers. Now, whether we add or subtract these several quantities, or multiply them by any whole number or divide them by any exact divisor, the

sums, differences, products, and quotients will also be integers. (See the Axioms.)

Suppose, then, we subtract the third from the second, and this remainder from the first. The result will be,

$$\frac{d' d'' x - d d'' x + d d' x - d' d'' r + d d'' r - d d' r}{d d' d''}, \text{ an integer.}$$

This will fulfil all the conditions. For, dividing by d, we obtain

$$-d'' x + d' x + d'' r - d' r + \frac{d' d'' x - d' d'' r}{d} = \frac{d' d'' (x - r)}{d};$$

the possibility of which depends, inasmuch as d, d', d'' are supposed to be prime to each other, upon the original supposition that $\frac{x-r}{d}$ is an integer.

In the same manner, it may be shown that all the other conditions will be fulfilled.

Rule. — Reduce the fractions $\frac{x-r}{d}$, $\frac{x-r'}{d'}$, $\frac{x-r''}{d''}$, &c., to equivalent fractions having a common denominator. Set the numerators in a column, one under another, the largest at the top, as follows: —

A	$d' d'' x - d' d'' r$,
B	$d d'' x - d d'' r'$,
C	$d d' x - d d' r''$, &c.

Subtract C from B, and this remainder from A. Thus bring all the numerators in the operation. Then continue subtracting these remainders, or some multiples or exact quotients of them, one from another, until x is obtained with unity for a coefficient; and rejecting the least common denominator from the number on the right as often as there is an excess. Then the last value on the right, if negative, is the answer sought. Otherwise it must be subtracted from the denominator, and the remainder will be the answer.

Example.

What number is that which, being divided by 11, 19, and 29, will leave the remainders 3, 5, and 10?

$\frac{x-3}{11}$, $\frac{x-5}{19}$, $\frac{x-10}{29}$, are integers,

or,

$\frac{551x-1653}{6061}$, $\frac{319x-1595}{6061}$, $\frac{209x-2090}{6061}$

$$\begin{array}{rr}
\text{A,} & 551x - 1653 \\
\text{B,} & 319x - 1595 \\
\text{C,} & \underline{209x - 2090} \\
\text{B}-\text{C,} & 110x + 495 \\
\text{A}-(\text{B}-\text{C}), & 441x - 2148 \\
4(\text{B}-\text{C}), & \underline{440x + 1980} \\
 & x - 4128
\end{array}$$

4128 = the answer.

The reason for taking 4128 as the value of x, as also for subtracting the number, when positive, from the common denominator, may be found by referring to the second method under the second Problem: the full expression in this case being $\frac{x-4128}{6061}$.

Problems of this kind may also be solved after reducing them to a common denominator, and combining the numerators in any manner thought best, according to the third method under Problem II.

Take, for illustration, from the previous example the quantity expressed by A — (B — C), $\frac{441x-2148}{6061} = y$. Then

$$\overset{a}{441}x - \overset{b}{6061}y = \overset{c}{2148}.$$

$$\begin{array}{r}
3 \\ 1 \\ 2 \\ 1 \\ 9 \\ 3 \\ \underline{1} \\ 4 \\ 37 \\ 41 \\ 119 \\ 160 = y' \\ 2199 = x'
\end{array}$$

Then, by formula 4, as the number of quotients is odd, we have

$$x = c(b - x') = 2148(6061 - 2199) = 8295576.$$

This is not the least value of x, but that value + some multiple of b. (See Prop. VIII.) We must therefore reject this mutiple of b; or, in other words, divide 8295576 by 6061, and there will remain 4128, as before.

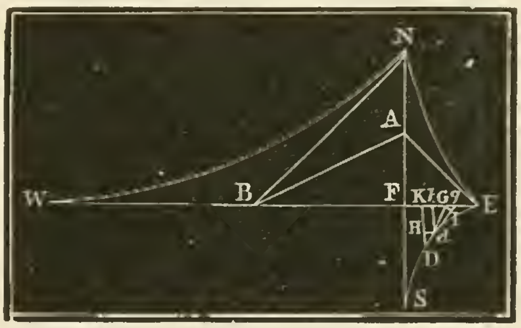
For this example, however, the former method is more concise.

SOLUTION OF A PROBLEM IN "CURVES OF PURSUIT."

By Dr. Joel E. Hendricks, Newville, Indiana.

Problem. — Three foxes are supposed to be at F, and a dog at S, forty rods south of F. They all start at the same instant, the foxes running with the same uniform velocity, and the dog just twice as fast. The first fox runs east, along the line FE; the second one runs north, along the line FA; and third runs west, along the line FW. The dog runs directly towards the first fox, and overtakes it at E. He immediately pursues the second one, and overtakes it at N; then pursues the third, and overtakes it at W. How far must the dog run to catch the three foxes? — Communicated by Prof. Root (Vol. I., p. 249).

Solution. — Let D represent the position of the dog when the first fox is at G, and draw DK parallel to SF, and dk infinitely near and parallel to DK. Also draw DG tangent to the curve at D, and dg tangent at the point d, and draw GI perpendicular to dg, and dH perpendicular to DK. Put $SF(=40)=a$, $FK=w$, $FG=x$, $DG=y$, and curve $SD=z$; and let the corresponding lines for the second and third curves be denoted by w_1, x_1, y_1, z_1, and w_2, x_2, y_2, z_2. Then is $Kk=dw$, $Gg=dx$, $Ig-Dd=dy$, and $Dd=dz$; and by the similar triangles HDd and IGg we have



$$dz : dw :: dx\left(=\tfrac{1}{2}dz\right) : \tfrac{1}{2}dw = Ig. \quad \therefore dy = \tfrac{1}{2}dw - dz.$$

But $\quad dz = 2\,dx, \quad \therefore dy = \frac{1}{2}dw - 2\,dx, \quad \therefore dx = \frac{1}{4}dw - \frac{1}{2}dy.$ Whence by integrating we have

$$(1) \qquad x = \tfrac{1}{4}w - \tfrac{1}{2}y + C.$$

Similarly,

(2) $x_1 = \frac{1}{4} w_1 - \frac{1}{2} y_1 + C_1.$ (3) $x_2 = \frac{1}{4} w_2 - \frac{1}{2} y_2 + C_2.$

When $w = 0$ in (1),

$$x = 0, \qquad y = a; \qquad \therefore\ C = \tfrac{1}{2} a.$$

When $w_1 = 0$ in (2),

$$x_1 = FE, \qquad y_1 = FE\sqrt{2}; \qquad \therefore\ C_1 = \tfrac{1}{2} FE(2 + \sqrt{2}).$$

When $w_2 = 0$ in (3),

$$x_2 = FN, \qquad y_2 = FN\sqrt{2}, \qquad \therefore\ C_2 = \tfrac{1}{2} FN(2 + \sqrt{2})^2.$$

By substituting for C, C_1, C_2, in (1), (2), (3), we have

(4) $$x = \tfrac{1}{4} w - \tfrac{1}{2} y + \tfrac{1}{2} a.$$

(5) $$x_1 = \tfrac{1}{4} w_1 - \tfrac{1}{2} y_1 + \tfrac{1}{2} FE(2 + \sqrt{2}).$$

(6) $$x_2 = \tfrac{1}{4} w_2 - \tfrac{1}{2} y_2 + \tfrac{1}{2} FN(2 + \sqrt{2}).$$

Now, when the dog overtakes the fox in running on each of the curves, the points D and G will coincide; and consequently y will become zero, and w will become x. Hence, if x', x_1', x_2' denote the whole distance run by the three foxes respectively, we shall have,

$$x' = \tfrac{1}{4} x' + \tfrac{1}{2} a; \qquad \therefore\ x' = \tfrac{2}{3} a.$$
$$x_1' = \tfrac{1}{4} x_1' + \tfrac{1}{3} a(2 + \sqrt{2}); \qquad \therefore\ x_1' = \tfrac{4}{9} a(2 + \sqrt{2}).$$
$$x_2' = \tfrac{1}{4} x_2' + \tfrac{2}{9} a(2 + \sqrt{2})^2; \qquad \therefore\ x_2' = \tfrac{8}{27} a(2 + \sqrt{2})^2.$$

By the question, $z' = 2x' = \frac{4}{3} a =$ 53.333

$$z_1' = 2(x_1' - x') = 2a\left(\tfrac{4}{9}(2 + \sqrt{2}) - \tfrac{2}{3}\right) = \tfrac{4}{9} a(1 + \sqrt{2}) = \quad 68.061$$
$$z_2' = 2(x_2' - x_1') = 2a\left(\tfrac{8}{27}(2 + \sqrt{2})^2 - \tfrac{4}{9}(2 + \sqrt{2})\right) = \tfrac{8}{27} a(6 + 5\sqrt{2}) = 154.916$$

Therefore the whole distance run by the dog = 276.310 rods.

Second Solution, by Prof. O. Root, Hamilton College, Clinton, N.Y.

Take the origin at F, and let xy be the co-ordinates of any point in the curve described by the dog in the pursuit; then will

$\int (dx^2 + dy^2)^{\frac{1}{2}}$ be the space passed over by the dog, and $x - \frac{y\,dx}{dy}$ that passed over by the first fox in the same time; but as these spaces are as 2 to 1, we have

$$x - \frac{y\,dx}{dy} = \tfrac{1}{2} \int (dx^2 + dy^2)^{\frac{1}{2}}.$$

By differentiating and reducing, the equation becomes

$$-\frac{d^2 x}{(dx^2 + dy^2)^{\frac{1}{2}}} = \tfrac{1}{2}\,\frac{dy}{y};$$

and by integrating,

$$\text{(1)} \qquad \log\left(\frac{dx}{dy} + \frac{(dx^2 + dy^2)^{\frac{1}{2}}}{dy}\right) = -\tfrac{1}{2}\log y + C.$$

If now we put $SF = 40$ rods $= a$, then, when $y = a$, we have $dx = 0$, and hence $C = \frac{1}{2}\log a$; and therefore (1) becomes

$$\log\left(\frac{dx}{dy} + \frac{(dx^2 + dy^2)^{\frac{1}{2}}}{dy}\right) = \log\frac{a^{\frac{1}{2}}}{y^{\frac{1}{2}}};$$

or,

$$\frac{dx}{dy} + \frac{(dx^2 + dy^2)^{\frac{1}{2}}}{dy} = \frac{a^{\frac{1}{2}}}{y^{\frac{1}{2}}};$$

or,

$$2\,dx = -\frac{a^{\frac{1}{2}}}{y^{\frac{1}{2}}}\,dy + \frac{y^{\frac{1}{2}}}{a^{\frac{1}{2}}}\,dy;$$

or, (2)

$$2x = -2a^{\frac{1}{2}}y^{\frac{1}{2}} + \tfrac{2}{3}\frac{y^{\frac{3}{2}}}{a^{\frac{1}{2}}} + C'.$$

When $y = a$, $x = 0$; $\therefore C' = \frac{4a}{3}$, and (2) becomes

$$2x = 2a^{\frac{1}{2}}y^{\frac{1}{2}} + \tfrac{2}{3}\frac{y^{\frac{3}{2}}}{a^{\frac{1}{2}}} + \frac{4a}{3},$$

which is the equation of the curve. When the dog overtakes the fox $y = 0$, and therefore $2x = \frac{4a}{3} = 53\frac{1}{3}$ rods for the distance run to catch the first fox.*

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* We omit the remainder of the solution, as the student will find no difficulty in completing it. It will be observed that Dr. Hendrick's solution involves a differential equation of only the first order, while Prof. Root's, which is the solution usually given, involves one of the second order. Is Dr. Hendrick's ingenious solution new? We have never seen it before. — Editor.

NOTE ON DIACAUSTICS.

By Rev. Thomas Hill, President of Antioch College, Yellow Springs, Ohio.

While acknowledging the justice of Mr. Paterson's remarks (Math. Monthly, p. 40, Vol. I.) on the waste of time involved in rediscovering that which was known centuries ago, I beg permission also to amuse myself with mathematical trifles even when I have not time to look up the results of previous investigators. The following note on Caustics is offered to the "Monthly," because it is interesting to me; but I do not know whether any or all of it is to be found in other places, with the exception of the corollary concerning the boundary of the space run through by a line in a rolling circle, which may be found, having no connection with caustics, in Chasles.

1. Let $\varrho = f\nu$ be the equation of any plane curve, ϱ being the radius of curvature, and ν the angle it makes with a fixed axis. Let θ be the angle which parallel rays of light make with the axis of ν.

2. If the distance from the curve to the caustic, measured on the reflected ray, be called d, we have, by trigonometry,

$$\frac{d}{\varrho\, d\nu} = \frac{\sin\left(\frac{1}{2}\pi - (\theta - \nu)\right)}{\sin 2\, d\nu},$$

(1) $$d = \tfrac{1}{2}\varrho \cos(\theta - \nu).$$

3. The element of the caustic is

$$d d + \varrho\, d\nu \sin(\theta - \nu) = \tfrac{1}{2} d\varrho \cos(\theta - \nu) + \tfrac{3}{2}\varrho \sin(\theta - \nu)\, d\nu;$$

and this divided by $2\,d\nu$ gives the radius of the caustic,

(2) $$\varrho_1 = \tfrac{1}{4}\left(D\varrho \cos(\theta - \nu) + 3\varrho \sin(\theta - \nu)\right).$$

4. From (1) it follows that a caustic is at its maximum distance from the generating curve, when the parallel rays strike the curve at right angles, the distance being half the radius of curvature of the curve.

5. From (2) it follows that the radius of a caustic, when parallel rays strike the generating curve at right angles, is one fourth the radius of the evolute of the curve. But where the rays strike the curve tangentially, the radius of the evolute is three fourths that of the curve itself.

6. Equation (2) may be reduced for any particular curve to the form $\varrho_1 = f\,\nu_1$, by putting

$$\nu = \tfrac{1}{2}(\nu_1 + \theta - \tfrac{1}{2}\pi).$$

7. For example, let $\varrho = 4\,R \sin \nu$; and we have

$$\begin{aligned}\varrho_1 &= R \cos^2 \nu \cos \theta + 4\,R \cos \nu \sin \nu \sin \theta - 3\,R \sin^2 \nu \cos \theta \\ &= -R \cos \theta + 2\,R \cos \theta \sin(\nu_1 + \theta) - 2\,R \sin \theta \cos(\nu_1 + \theta) \\ &= 2\,R \sin \nu_1 - R \cos \theta.\end{aligned}$$

That is to say, the caustic of a cycloid, for parallel rays, is concentric with a cycloid of half the radius, coinciding with the small cycloid when $\theta = \frac{1}{2}\pi$.

8. Let the circle generating the original cycloid be again rolled through it, and it is manifest that the reflected ray will make, with a diameter through the generating point, the constant angle $\frac{1}{2}\pi - \theta$. Hence a diameter of the generating circle is a sliding tangent on the small cycloid, and any chord in the circle is a sliding tangent on a curve concentric with a half-size cycloid.

9. The caustic for $\varrho = A \sin^n \nu$ is (see Gould's Astr. Journal),

$$\varrho_1 = \tfrac{1}{4} A\,(n \sin^{n-1} \nu \cos(\theta - \nu) \cos \nu + 3\,A \sin^n \nu \sin(\theta - \nu))\,;$$

which, by putting $\theta = \frac{1}{2}\pi$, reduces to

$$(3)\quad \varrho_1 = \frac{3+n}{4} A \sin^n \nu \cos \nu = \left(\frac{1}{2}\right)^{\frac{n+1}{2}} \left(\frac{3+n}{4}\right) A\,(1 - \cos \nu_1)^{\frac{n}{2}} (1 + \cos \nu_1)^{\frac{1}{2}}.$$

10. The substitution in the last equation of $n = 1$ gives for the cycloid, as in Art. 7,

$$\varrho_1 = \frac{A}{2} \cdot \sin \nu_1.$$

The substitution of $n = -3$ gives for the parabola a single point, $\varrho_1 = 0$.

11. Equation (3) may also be written,

$$(4) \qquad \varrho_1 = \frac{3+n}{4} A \sin^n \tfrac{1}{2} \nu_1 \cos \tfrac{1}{2} \nu_1.$$

12. The integral of (4) gives, for the involute of the caustic,

$$\varrho_2 = \frac{3+n}{2(n+1)} A \sin^{n+1} \tfrac{1}{2} \nu_1 = A_1 \sin^m \tfrac{1}{2} \nu_1.$$

13. The last equation shows that the involutes of the caustics of $\varrho = A \sin^n \nu$ have the same form as the original curves, except that $\frac{1}{2} \nu$ is substituted for ν, and 1 is added to n. The caustics bear the same relation to the evolutes of the curves.

14. The substitution of $\frac{1}{2} \nu$ for ν in the equation of a curve will make it necessary for ϱ to revolve twice as far before going through the same changes. This may be conceived as effected by either of the following metamorphoses.

Conceive the curve as composed of infinitesimal links, in which $d\nu$ is of equal value. Imagine these links capable of extension or retraction without altering their curvature, and it will be evident that by making them each a times as long as before, ν will change a times as much for a given change of ϱ, so that the equation will be changed from $\varrho = f\nu$ to $\varrho = f\frac{\nu}{a}$. The extension may be conceived as effected by sliding joints.

Or imagine these links capable of being bent to a greater or less curvature, their length being unchanged; and it will be evident that by giving each link b times as much curvature, ν must change b times as much in order to produce the same order of changes in ϱ, and at the same time ϱ will be diminished to $\frac{\varrho}{b}$. That is to say, $\varrho = f\nu$ will become $\varrho = \frac{1}{b} f \frac{\nu}{b}$.

Thus the first change will convert the cycloid $\varrho = 4R \sin \nu$ into the epicycloid $\varrho = 4R \sin \frac{\nu}{a}$, while the second change would convert the same cycloid into the epicycloid $\varrho = \frac{4a}{b} \sin \frac{\nu}{b}$. Hence the

cycloid, by sliding into itself, or by extending itself, or by having the curvature of each part increased or diminished in proportion to the existing curvature, may be changed into any of the epicycloids and hypocycloids $\varrho = A \sin a\,\nu$, in which $\frac{A\,a}{1 - a^2}$ is the radius of the stationary, and $\frac{A}{2\,(1 + a)}$ that of the generating.

15. Other examples of the peculiar deformation to which the caustics of $\varrho = A \sin^n \nu$ have led us, are reserved for another note.

ON THE LOCUS OF PERPENDICULAR TANGENTS TO ANY CONIC SECTION.

By William Chauvenet, Professor of Mathematics in Washington University, St. Louis.

"Pairs of tangents perpendicular to each other, are drawn to any conic section; find the locus of their intersection."

The solution of this problem, where the given conic section is the ellipse, is well known. It was proposed and solved in a very elegant manner by Bessel, in the introduction to the volume of the Königsberg Observations for 1814, where it is applied to determining the effect of an elliptical form of the pivot of a transit instrument revolving in a V, the faces of which are perpendicular to each other. To obtain a general solution, I take the equation

$$y^2 = 2px + x^2,$$

which represents any conic section; the axis of x being the axis of the curve, the origin being the extremity of the axis of the curve (or vertex), $2p$ being the parameter, and n depending on the nature of the curve. If now a', a'' denote the trigonometric tangents of the angles which two tangent lines to the given curve make with the axis of x, (x', y') and (x'', y'') the points of contact, we have, for the equations of these tangent lines,

$$(1) \qquad -a'\,(x - x') + (y - y') = 0,$$

$$(2) \qquad -a''\,(x - x'') + (y - y'') = 0,$$

with the conditions,

$$(3) \qquad a'a'' + 1 = 0,$$

$$(4) \qquad a'y' = p + nx',$$

$$(5) \qquad a''y'' = p + nx'',$$

$$(6) \qquad y'^2 = 2px' + nx'^2,$$

$$(7) \qquad y''^2 = 2px'' + nx''^2;$$

of which (4) and (5) are obtained by differentiating our general equation and putting a' and a'' for $D_{x'}(y')$ and $D_{x''}(y'')$.

From these seven equations we have to eliminate the six quantities $x', y', x'', y'', a', a''$, to obtain the equation of the required locus involving only x and y (which, being common to the two tangents, will represent their intersection), and the constants n and p.

Substituting in (1), (2), (6), and (7), the values of x' and x'', given by (4) and (5), we find

$$(8) \qquad (a'^2 - n)\,y' = a'nx - ny + a'p,$$

$$(9) \qquad (a''^2 - n)\,y'' = a''nx - ny + a''p,$$

$$(10) \qquad (a'^2 - n)\,y'^2 = p^2,$$

$$(11) \qquad (a''^2 - n)\,y''^2 = p^2;$$

from which we immediately deduce

$$(12) \qquad (a'nx - ny + a'p)^2 - (a'^2 - n)\,p^2 = 0,$$

$$(13) \qquad (a''nx - ny + a''p)^2 - (a''^2 - n)\,p^2 = 0,$$

or,

$$(14) \qquad n\,(a'x - y)^2 + 2\,a'p\,(a'x - y) + p^2 = 0,$$

$$(15) \qquad n\,(a''x - y)^2 + 2\,a''p\,(a''x - y) + p^2 = 0.$$

Multiplying (15) by a'^2, and observing the condition (3), we have

$$n\,(x + a'y)^2 + 2p\,(x + a'y) + a'^2p^2 = 0;$$

which, added to (14), gives

$$(16) \qquad (nx^2 + ny^2 + 2\,px + p^2)\,(a'^2 + 1) = 0,$$

in which the factor $a'^2 + 1$ cannot be zero; and hence we must have

(A) $$nx^2 + ny^2 + 2px + p^2 = 0,$$

which is the equation of the required locus. If put under the form

(B) $$\left(x + \frac{p}{n}\right)^2 + y^2 = \frac{p^2}{n^2}(1 - n) = R^2,$$

it shows that the locus is in general the circumference of a circle, the centre of which is in the axis at the distance $-\frac{p}{n}$ from the origin, and whose radius is $R = \frac{p}{n}\sqrt{1+n}$.

For the ellipse, the semi-axes of which are A and B, we have

$$n = -\frac{B^2}{A^2}, \qquad -\frac{p}{n} = A, \qquad R = \sqrt{A^2 + B^2};$$

so that the centre of the circle is the centre of the ellipse, and the diameter of the circle is the diagonal of the rectangle continued by the axes of the ellipse.

For the hyperbola, the semi-axes of which are A and B, we have

$$n = +\frac{B^2}{A^2}, \qquad -\frac{p}{n} = -A, \qquad R = \sqrt{A^2 + B^2};$$

and the centre of the circle is still the centre of the curve. But if $A < B$ there is no locus, since R is then imaginary. If $A = B$, $R = 0$, and the locus becomes a point which is the centre of the hyperbola. That is, in the equilateral hyperbola only one pair of tangents perpendicular to each other can be drawn, and these are evidently the asymptotes.

For the parabola, we have

$$n = 0, \qquad -\frac{p}{n} = \infty, \qquad R = \infty;$$

so that the locus is a straight line whose equation is, by (A),

$$x = -\frac{p}{2},$$

which is the equation of the directrix of the parabola.

This solution may interest the student as an example of elimination, the "method of substitution" leading directly to the required result.

THE CENTRAL SCHOOL OF ARTS AND MANUFACTURES AT PARIS.

[It gives us great pleasure to call attention to the following communication from MR. WATSON. His notes on the Polytechnic and other Schools of Paris, will be given in early numbers of Vol. III.]

HISTORY, GENERAL PLAN, CONDITIONS FOR ADMISSION, COURSES OF LECTURES, DRAWINGS, MANIPULATIONS IN THE LABORATORIES OF CHEMISTRY AND PHYSICS, DISCIPLINE, ETC.

PECLET, DUMAS, and OLIVIER, founded the Central School in 1829. PECLET, a graduate of the Normal School, had been for some time a Professor of Physics at Marseilles; after the restoration, on account of his political opinions, he had lost his place, yet he still devoted himself to the industrial applications of his science.

DUMAS, at that time a Repetiteur of Chemistry in the Polytechnic School, was already known by important scientific works. OLIVIER, also a graduate of the Polytechnic School, had just returned from Sweden, where he had reorganized the schools of Artillery, and founded a chair of Descriptive Geometry, applied to military service. These three men, theorists by their studies, had become practical from the nature of their works. They selected the name of Central School of Arts and Manufactures for two reasons, first because it had been the former name of the Polytechnic School, which they wished simply to re-establish according to the original design of the founders, and secondly that no one should be deceived as to their own intention. They framed for it the following plan, which is rigidly adhered to in every particular at present.

The basis of the teaching must be the study of Geometry, Mechanics, Physics, and Chemistry (since every engineer should possess an extended knowledge of these four sciences); and the teaching of them must always be conducted with a view to their application to industry. This will form the instruction for the first year, which is to be preparatory, and during which the student is fitted to enter upon the courses of the second and third years, which form, so to speak, the school of application. The professors must never forget that the object of the school is *to form instructed and capable engineers, and not savans.*

They will exhibit in detail, during the first year, those theories leading to the applications in industry, and for all the others, they will expose them summarily, so that the students may know what has been found out, and may know where they can study them in detail.

The professors of the first year must not be purely savans, but must be taken from the engineers, retired or in active service: in a word they should be, without ceasing, in contact with industry, and by profession or by taste should love or cultivate the applications of these sciences.

During the first year the students will execute drawings in Descriptive Geometry in all its applications, and also in Mechanics and Architecture.

They will make manipulations in Chemistry and Physics, the practice being always annexed to the oral exposition.

In the courses of the second and third years, the students will be shown the applications of the theories of the first year to different works of engineering.

The students will follow all the courses, but will only execute projects and manipulations in their speciality.

All the professors of the courses of the second and third years *must* be engineers, for it is only the man who has practised (" *Qui a mit la main a la pâte* ") who is able to teach all the details of the construction of a bridge, of a water-wheel, of a railroad, of a steam-engine, in the best manner, and it also accomplishes the object of keeping the school up to the level of the requirements of industry. Young men are admitted between the ages of seventeen and thirty, from all countries, of all political opinions, and of every religion: by this means a desirable fusion is

established between the different political and religious parties which will eventually lead the way to industrial accommodations between the different nations.

The school in 1857 became an establishment of the state, but no modification has been made, unless it has been to raise a little the standard for graduation.

The following are the requisites for admission this present year (1860): —

1. The French language. 2. Arithmetic. 3. Geometry, plane and solid. 4. Algebra, to the general theory of equations. 5. Plane Trigonometry. 6. Analytical Geometry. 7. The elements of Descriptive Geometry. 8. General notions of Physics. 9. Inorganic Chemistry. 10. Natural History. 11. The elements of Anatomy and Physiology, Zoölogy and Botany. 12. Design, in which each student will present a collection of drawings relative to Descriptive Geometry and the tracing of curves of the second degree; also, some Architectural drawings shaded in India ink, and also a book of sketches of machines.

The applicants for admission are examined in writing, during which they have to make a drawing in Descriptive Geometry, and also one in Architecture, after a model. They are also afterward examined orally. Some idea of the labor and thoroughness of the examination will be formed from the fact that last year eight hundred persons were examined for admission, and of these only two hundred and fifty were admitted, the examination commencing the first of August and terminating the fifteenth of October.

The courses of the school commence the first of November, and last, including the examinations, until the middle of August. The students do not lodge in the establishment, but go every day before 8 ½ A. M., and remain until 4 P. M. All this time, except one hour for breakfast, is devoted to work of some kind, and during this time they are under the direction of the professors and inspectors.

There are attached to the school, sixteen professors, eleven repetiteurs, and two inspectors.

The duty of the professor is to give the instruction by lectures entirely, excepting such as must necessarily be given to each pupil separately, as in drawing and constructions.

The duty of each repetiteur is to examine and mark the students upon the subjects of the lectures of the professors, each student is examined in some course once every week, and *alone*, at which time he is obliged to show to the repetiteur the notes taken upon the subjects upon which he is examined.

The courses of Lectures for the first year are: —

1. Descriptive Geometry. 2. Physics. 3. Mechanics. 4. Chemistry. 5. Natural History. 6. Cinematics, or (Transformation of movements in machines).

The course of Descriptive Geometry comprehends the two treatises of Leroy on Descriptive Geometry and Sterotomie, in its Applications to Shades and Shadows, Linear Perspective, Dialing, Stone-cutting, and Carpentry, and besides the drawings of all of these, the students are required to make models of the different arches and domes, from chalk, and models in wood of the various details in Carpentry.

The students have two lectures in each course per week.

In the course of Physics, the students have such problems as this to solve. Determine the specific gravity of an alloy, taking into account the temperature of the water, and the barometric pressure.

In Chemistry each student makes twenty manipulations, of three hours each, from the preparation of oxygen to the preparation of the ethers.

The course on Mechanics commences with Analytic Geometry, and includes the Elements of the Differential and Integral Calculus, and also Theoretical Mechanics.

The students have problems to work. Here is one in Mechanics. A material point moves in a plane, and the laws of the velocities along two axes at right angles are known.

Deduce 1st, the co-ordinates of the moving point in functions of the time. 2d, Determine its

velocity. 3d, Calculate the acceleration along each axis, and thence the total, and also the tangential and normal acceleration. 4th, Discuss the curve in which the body moves, when

$$V_x = A\,(1 + e^{-wt}), \qquad A = 1, \qquad w = 0.004.$$
$$V_y = (a + b\,t), \qquad a = 2.4, \qquad b = 0.12.$$

The two courses of Natural History, and that of Cinematics, last five months each.

During vacation the students are required to make two drawings, one Architectural, and one of a machine ; these are left to the selection of the student.

At the beginning of the second year each student chooses one speciality ; these are four in number, viz. that of a Metallurgist, Constructor, Chemist, and Machinist. They have to follow all the courses but only execute projects and manipulations in their own speciality.

The courses are the same for the second and third years, except that a course upon Railroads is added the third year ; they are as follows : — 1. Applied Mechanics. 2. Industrial Physics. 3. Industrial Chemistry. 4. Metallurgy. 5. *Exploitation* of Mines. 6. Analytical Chemistry. 7. Construction. 8. Construction of Machines.

Here is a project upon Applied Physics. To make the plans with all the details for the construction of a steam-boiler, which shall produce 700 kilogrammes of steam per hour.

Here is a project in the Construction of Machines. Construct a steam-engine to raise by means of pumps, the water to feed a reservoir at a railway station. The dimensions of the reservoir are 200 m. cube, the bottom of the reservoir is 6 m. above the ground. A noncondenser is required which shall work one or many pumps ; the level of the water is 15 m. below the ground, and the amount required per hour is 33 m. cube.

In Construction projects are given, as the construction of a truss bridge of given dimensions. Or the construction of a viaduct in masonry. These projects are prepared in detail, and are often the work of a month or more of hard labor, consisting, as they do, of five or six large drawings and many pages of memoirs.

At the end, after a severe examination, if the student has passed successfully, he receives the diploma of civil engineer, without which he cannot practise in this country.

Discipline. Every student is obliged to be under the care of his parents or their correspondent; so that if there is any misconduct on the part of the student, his correspondent is immediately notified. Whenever the student is absent from school the correspondent is notified of that fact the same day. There are private and public reprimands given by the director; and in case the pupil is refractory, the correspondent is invited to remove him. I have been thus minute in describing the school for two reasons, that those young men who intend to finish their engineering education may know what sort of a school it is, and what they have to submit to, and also, that teachers may see that in the amount to be taught, if not in the methods, there is room for some improvement.

This is not by any means considered as equal in the theoretical instruction to the Polytechnic School, and its graduates cannot have that complete, and special, and practical skill, which the engineers trained in the Polytechnic, and afterward in the School of Mines or the École des Ponts et Chaussées. Of each one of these I shall have occasion to speak in detail.

I must not omit to say that the price of instruction in the Central School is 775 francs per year, and that the government has established a number of scholarships which are given to meritorious students ; also, that the students, accompanied by their Professor of Design, make excursions to some edifice, or bridge, or manufactory in Paris, and make sketches and measurements of them on the spot, and afterwards draw them in detail. They also take excursions to make geological exploration in the neighborhood of Paris.

Paris, August 6, 1860.

Editorial Items.

We have received the following solutions of the Prize Problems in the June number of the Monthly: —

Harriet S. Hazeltine, Worcester, Mass., Probs. I., II.

George B. Hicks, Cleveland, Ohio, Prob. III.

David Mercer, Okeana, Ohio, Probs. I., II.

W. T. Jackson, Western College, Linn Co., Iowa, Prob. II.

Asher B. Evans, Madison University, N. Y., Probs. I., II., III., IV., V.

O. B. Wheeler, Michigan University, Ann Arbor, Probs. III., IV., V.

Joseph B. Davison, Oberlin College, Ohio, Probs. I., II., III.

Gustavus Frankenstein, Springfield, Ohio, Prob. IV.

We have delayed the report of the Judges until our next number, in order to complete the article "On the Indeterminate Analysis" in this volume.

Prize Essays. — We have received four Prize Essays, and regret that Prof. Ferrel's absence has prevented the committee from giving their report in this number; but we hope to be able to announce the decision in our next.

Errata. — Page 108, for "$b^2 - d^2$" or "$b^2 - a^2$," read "$d^2 - b^2$"; page 181, line 16, "$x - y$" should precede "x"; page 329, Prob. III., for "tan 245°," read "tan 254°"; page 374, for "$t = 90°$," read "$t = - 90°$."

Indexes. — Our special thanks are due to our friend and colleague, E. J. Loomis, Esq., for the Indexes to this volume.

The Illustrated Pilgrim Almanac for 1861. Published in Aid of the Monument Fund. Boston: Published at the Office of the National Monument to the Forefathers, No. 13 Tremont Row. Willard M. Harding, Financial Agent.

Most of our readers are probably already aware that the first number of this Almanac was for the year 1860, and published in August, 1859. The success of the first number was such as to warrant its continuance, and to demonstrate that the publication will largely aid the Monument Fund, not only directly, but by increasing public interest in the enterprise. It will be praise enough for the illustrations in the numbers before us, to say that they are by Hammatt Billings, Esq., the designer of the Pilgrim Monument.

The heads of the calendar pages in the number for 1860 represent several of the celebrated monuments of ancient and modern times; for the year 1861 they represent the discoveries of America, Florida, the Mississippi, St. Lawrence, Expedition of Sir Walter Raleigh, First Settlement of Virginia, Discovery of Hudson River, Settlement of Rhode Island, First Sabbath in New Haven, Settlement of Pennsylvania and Kentucky, and Emigration in the United States. Besides these, we find the Breakwater at Plymouth, England, Attempts of the Pilgrims to escape to Holland, the Pilgrim Meeting-House on Burying Hill, Plymouth, Capture of Annawan, and Immigration into the United States. The number closes with the names of 3,412 members of the Pilgrim Society who have subscribed $ 5, or upwards.

The calendars are given for five different parallels of latitude, — namely, Montreal, Boston, New York City, Washington, and Charleston, — thus making the work available for all parts of the United States and Canadas.

The work is beautifully printed on fine tinted paper, and is as ornamental as useful; and, at the low price of twenty-five cents a number, must have a large and steadily increasing sale.

INDEX OF PROPER NAMES.

Names of Contributors are in Small Capitals.

ALPHABETICAL INDEX.